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**Quantum Groups, Roots of Unity
and
Particles on quantized Anti-de Sitter Space ¹**

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Abstract

Quantum groups in general and the quantum Anti-de Sitter group $U_q(so(2, 3))$ in particular are studied from the point of view of quantum field theory. We show that if q is a suitable root of unity, there exist finite-dimensional, unitary representations corresponding to essentially all the classical one-particle representations with (half)integer spin, with the same structure at low energies as in the classical case. In the massless case for spin ≥ 1 , the "naive" representations are unitarizable only after factoring out a subspace of "pure gauges", as classically. Unitary many-particle representations are defined, with the correct classical limit. Furthermore, we identify a remarkable element Q in the center of $U_q(g)$, which plays the role of a BRST operator in the case of $U_q(so(2, 3))$ at roots of unity, for any spin ≥ 1 . The associated ghosts are an intrinsic part of the indecomposable representations. We show how to define an involution on algebras of creation and annihilation operators at roots of unity, in an example corresponding to non-identical particles. It is shown how nonabelian gauge fields appear naturally in this framework, without having to define connections on fiber bundles. Integration on Quantum Euclidean space and sphere and on Anti-de Sitter space is studied as well. We give a conjecture how Q can be used in general to analyze the structure of indecomposable representations, and to define a new, completely reducible associative (tensor) product of representations at roots of unity, which generalizes the standard "truncated" tensor product as well as our many-particle representations.

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Introduction

The topic of this thesis is the study of quantum groups and quantum spaces from the point of view of Quantum Field Theory.

The motivation behind such an endeavour is easy to see. Quantum field theory (QFT) is a highly successful theory of elementary particles, with an embarrassing "fault": except for some special cases, it cannot be defined without some sort of a regularization of the underlying space, which at present is little more than a recipe to calculate divergent integrals. Physically, it is in fact expected that space-time will not behave like a classical manifold below the Planck scale, where quantum gravity presumably modifies its structure. The hope is that this somehow provides a regularization for QFT. From a mathematical point of view, it also seems that there should exist some rigorous theory behind such "naive" quantum field theories, given the rich mathematical structures apparently emerging from them. And of course, it would be highly desirable to put the physically relevant quantum field theories on a firm theoretical basis.

In view of this, it seems very natural that a consistent theory of elementary particles should not be based on concepts of classical geometry, but rather on some kind of "fuzzy", or "quantized" space-time. With very little experimental guidance, finding the correct description may seem rather hopeless. The approach we will pursue here relies heavily on mathematical guidance, given the "unreasonable usefulness of mathematics in physics" (Wigner).

With the development of Non-Commutative Geometry in recent years [8], a possible candidate for a new framework has emerged. It is a generalization of the (rather old) idea that a manifold \mathcal{M} can be described by the algebra of functions $Fun(\mathcal{M})$ on it. In Non-Commutative Geometry, one considers instead some non-commutative algebras replacing $Fun(\mathcal{M})$, with sufficiently rich additional structures. A "quantum deformation" or simply "deformation" of a classical manifold is essentially a (non-commutative) algebra with a deformation parameter \hbar , such that the classical algebra

of functions on the manifold is obtained in the limit $\hbar \rightarrow 0$.

This idea is in fact very familiar to physicists: Quantum Mechanics can be viewed as a noncommutative geometry on phase space, and the Planck constant plays the role of the deformation parameter. This example also shows that while the limit $\hbar \rightarrow 0$ may be smooth in some sense, the physical interpretation may be very different. Furthermore if \hbar is dimensional, it is expected that the "quantum" case should behave classically at large enough scales.

This is a new and vast field of research, and to make any progress, one clearly has to choose a particular approach. A simple example of a theory of elementary particles on a noncommutative space was proposed by Connes [9] and Connes & Lott [10]. It is based on the space $M \times \mathbb{Z}_2$, where M is ordinary Minkowski space and \mathbb{Z}_2 is considered as a noncommutative space with a connection, which can be interpreted as a Higgs field. This leads to a new approach to the standard model. Fröhlich and collaborators [5] introduced gravity in this context.

Incidentally, it has been pointed out that string theory, a candidate for a theory underlying quantum field theory including gravity, seems to predict some noncommutativity of certain coordinate algebras [60]. Other recent developments [3] also suggest some relevance of Non-Commutative geometry to M-theory or string theory, which is traditionally formulated in the language of classical geometry.

Of course one would really like to consider truly noncommutative spaces. There have been many approaches to "quantize" physically relevant spaces like Minkowski space, fiber bundles, and many others. While many interesting examples have been found, a clear guideline is missing.

At this point, we want to emphasise (without the need to do so) the importance of Lie groups in elementary particle physics, notably the Poincare group which dictates the behaviour of free particles, and internal symmetry groups which may strongly constrain their interactions.

Quantum groups are remarkable examples of Noncommutative Geometry, since they can be viewed as deformations of classical Lie groups resp. their manifolds. Their mathematical structure is well studied and even richer than that of classical Lie groups. They depend on a deformation parameter $q = e^{\hbar}$, where $\hbar = 0$ corresponds to the classical case. Furthermore, they act naturally on associated quantum spaces.

Thus it seems that quantum groups, which combine the features of both Lie groups and Non-Commutative Geometry in an analytic way, should be a powerful guide towards a realization of the above ideas. In this thesis, we want to follow this

approach and see where it leads to.

It is fair to say that quantum groups are analytic "deformations" of classical groups, for "generic" q . However when q is a root of unity, their structure is in many ways very different, and one is facing a truly new and very rich mathematical object. One of the main points we want to make is that the root of unity case seems to be the most interesting one from a QFT point of view, beyond its known relevance to Conformal Field Theory [1, 40].

We will mainly study the Anti-de Sitter group $SO_q(2,3)$ and its representation theory, which will play the role of the Poincare group. This choice is vindicated by its simplicity and the wealth of interesting features found, which constitute the main part of this thesis. In the classical case, the Poincare group can be obtained from $SO(2,3)$ by a contraction; however we will not do a corresponding contraction in the quantum case [36], since our main results would all break down.

This thesis is organized as follows. Chapter 1 is a brief, general introduction to quantum groups and their representation theory, with emphasis on those aspects which will be important later. Whenever possible, a short explanation is included on how these facts can be obtained. We will mainly work with the "quantized universal enveloping algebra" $U_q(g)$. Most of this chapter is well known, but it also contains some new results and definitions.

In chapter 2, we consider quantum spaces associated to quantum groups, and study integration on quantum Euclidean space, sphere, and on quantum Anti-de Sitter (AdS) space. We point out that quantum AdS space has an intrinsic length scale, above which it looks like a classical manifold.

Chapter 3 starts with a brief review of the unitary representations of the classical AdS group corresponding to elementary particles, as well as a discussion of massless particles, (abelian) gauge theories and BRST from a group theoretic point of view. We then study these issues for the quantum AdS group. In particular, we show that for suitable roots of unity q , there are *finite-dimensional*, unitary representations corresponding to all the classical ones, with the same structure at low energies³. In the massless case for $\text{spin} \geq 1$, the "naive" representations contain a subspace of "pure gauges" which must be factored out to get unitary, irreducible representations, as classically. A definition of unitary many-particle representations is given. Furthermore, we identify a remarkable element Q of the center of $U_q(g)$ which plays the role of a (abelian) BRST operator in the case of the AdS group at roots of unity, for

³For the singleton representations, this was already shown in [12].

any spin ≥ 1 . The corresponding ghosts are an intrinsic part of the indecomposable representations.

In chapter 4, we give a conjecture that Q can be used for any group to understand the structure of the tensor product at roots of unity, and to define a new, associative (tensor) product of irreducible representations at roots of unity, which generalizes the well-known "truncated" tensor product used in conformal field theory, as well as the many-particle representations mentioned above. We then show how one can define an involution on algebras of creation and annihilation operators, for q a root of unity; lacking a symmetrization postulate at present, we have to work with a version corresponding to non-identical particles. It is shown how all this might be used towards constructing a quantum field theory. The main missing piece to achieve this goal is a way to define identical particles, i.e. a symmetrization postulate. Finally, we point out that nonabelian gauge fields appear naturally in this framework, without having to define something like connections on fiber bundles.

Chapter 1

Quantum Groups

1.1 Hopf Algebras and Quantum Groups

In this thesis, we will be concerned with the representation theory of quantum groups in the Drinfeld–Jimbo formulation, which is a certain Hopf algebra with additional structure to be explained below. The most economical approach to this goal would be to start with these given mathematical objects, and study its properties from the point of view we have in mind. However a reader who is not very familiar with quantum groups would be left in the dark wondering where all this comes from, and quite possibly develop some misconceptions. Therefore we first give a brief review of the underlying mathematical structure. For more details, the reader is referred to [15, 18] or a number of existing reviews, such as [6, 25].

There are at least two ways to introduce quantum groups. One is to consider the universal enveloping algebra of a simple Lie group, and discover a new structure on it, namely that of a quasitriangular Hopf algebra. The other, more geometric approach is to “quantize” the space of functions on a (compact) Lie group, which turns out to have a remarkable Poisson–structure. These two approaches are dual to each other, and they originated in the study of certain integrable models.

1.1.1 Hopf Algebras

The mathematical language to describe both points of view is that of a Hopf algebra. The most familiar example of a Hopf algebra \mathcal{A} is the space of functions $Fun(G)$ on a (compact) Lie group G . This is a *commutative* algebra by pointwise multiplication, but this has nothing to do with the group structure. The group multiplication is

encoded in \mathcal{A} as a *coproduct*, which is a map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, where $(\Delta(f))(x, y) = f(x \cdot y)$ for $f \in \text{Fun}(G)$. The inverse is encoded as *antipode* $S : \mathcal{A} \rightarrow \mathcal{A}$ where $(Sf)(x) = f(x^{-1})$ in the case of $\text{Fun}(G)$, and the unit element $e \in G$ becomes the *counit* $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$, with $\epsilon(f) = f(e)$ for $\text{Fun}(G)$. In this way, all the structure of G has been encoded in $\text{Fun}(G)$. In general, a Hopf algebra is an algebra \mathcal{A} with coproduct, antipode and counit and the following compatibility conditions:

$$(\Delta \otimes \text{id})\Delta(a) = (\text{id} \otimes \Delta)\Delta(a), \quad (\text{coassociativity}), \quad (1.1)$$

$$\cdot(\epsilon \otimes \text{id})\Delta(a) = \cdot(\text{id} \otimes \epsilon)\Delta(a) = a, \quad (\text{counit}), \quad (1.2)$$

$$\cdot(S \otimes 1)\Delta(a) = \cdot(\text{id} \otimes S)\Delta(a) = 1\epsilon(a), \quad (\text{coinverse}), \quad (1.3)$$

$$\Delta(ab) = \Delta(a)\Delta(b), \quad (1.4)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad \text{and} \quad (1.5)$$

$$\Delta(1) = 1 \otimes 1, \quad \epsilon(1) = 1, \quad (1.6)$$

for all $a, b \in \mathcal{A}$. This implies

$$S(ab) = S(b)S(a), \quad (\text{antihomomorphism}), \quad (1.7)$$

$$S(1) = 1, \quad (1.8)$$

$$\Delta(S(a)) = \tau(S \otimes S)\Delta(a), \quad \text{with} \quad \tau(a \otimes b) \equiv b \otimes a, \quad (1.9)$$

$$\epsilon(S(a)) = \epsilon(a). \quad (1.10)$$

We will use Sweedler's [54] notation for the coproduct:

$$\Delta(a) \equiv a_{(1)} \otimes a_{(2)} \quad (\text{summation is understood}). \quad (1.11)$$

\mathcal{A} is not required to be commutative, and in general it is non-cocommutative, i.e. $\Delta' \equiv \tau \circ \Delta \neq \Delta$.

1.1.2 $U_q(g)$ and Quasitriangular Hopf Algebras

The fastest way to introduce quantum groups is to simply write down a certain deformation of the universal enveloping algebra of a simple Lie algebra g in a Chevalley basis with a complex parameter q , and study its properties. We will mainly work in this framework, which was introduced by Drinfeld [15] and Jimbo [27].

Let $q \in \mathbb{C}$ and $A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ be the Cartan matrix of a classical simple Lie algebra g of rank r , where $(,)$ is the Killing form and $\{\alpha_i, \quad i = 1, \dots, r\}$ are the simple roots.

Then the *quantized universal enveloping algebra* $\mathcal{U} \equiv U_q(g)$ is the Hopf algebra with generators $\{X_i^\pm, H_i; \quad i = 1, \dots, r\}$ and relations [18, 27, 15]

$$[H_i, H_j] = 0 \quad (1.12)$$

$$[H_i, X_j^\pm] = \pm A_{ji} X_j^\pm, \quad (1.13)$$

$$[X_i^+, X_j^-] = \delta_{i,j} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{i,j} [H_i]_{q_i} \quad (1.14)$$

$$\sum_{k=0}^{1-A_{ji}} \begin{bmatrix} 1 - A_{ji} \\ k \end{bmatrix}_{q_i} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-A_{ji}-k} = 0, \quad i \neq j \quad (1.15)$$

where $d_i = (\alpha_i, \alpha_i)/2$, $q_i = q^{d_i}$, $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q_i} = \frac{[n]_{q_i}!}{[m]_{q_i}! [n-m]_{q_i}!}. \quad (1.16)$$

The last of (1.15) are the deformed Serre relations. As algebra, it can be shown [16] that if q is considered as a formal variable, \mathcal{U} is essentially¹ the same as the classical, undeformed enveloping algebra with a formal variable q and a different choice of generators. However it is *not* equivalent as Hopf algebra: the comultiplication on \mathcal{U} is defined by

$$\begin{aligned} \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i \\ \Delta(X_i^\pm) &= X_i^\pm \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_i^\pm, \end{aligned} \quad (1.17)$$

and antipode and counit are

$$\begin{aligned} S(H_i) &= -H_i, \\ S(X_i^+) &= -q^{d_i} X_i^+, \quad S(X_i^-) = -q^{-d_i} X_i^-, \\ \epsilon(H_i) &= \epsilon(X_i^\pm) = 0. \end{aligned} \quad (1.18)$$

The classical case is obtained by taking $q = 1$. The consistency of this definition can be checked explicitly.

The Cartan–Weyl involution is defined as

$$\theta(X_i^\pm) = X_i^\mp, \quad \theta(H_i) = H_i, \quad (1.19)$$

¹without going into mathematical details here

extended as a linear anti-algebra map (involution). In particular, $\theta(q) = q$ for any $q \in C$. It is obviously consistent with the algebra, and one can check that

$$(\theta \otimes \theta)\Delta(x) = \Delta(\theta(x)), \quad (1.20)$$

$$S(\theta(x)) = \theta(S^{-1}(x)). \quad (1.21)$$

The conventions we use are those of [18] except for a replacement $q \rightarrow q^{-1}$ for reasons explained in the next section, and agree up to normalization (see below) with those of [15, 33]. They differ from e.g. [6] by some redefinitions. In the mathematical literature, usually a rational version of the above algebra, i.e. using $q^{d_i H_i}$ instead of H_i is considered. Since we are mainly interested in specific representations, we prefer to work with H_i . Furthermore, if q is a root of unity, one has to specify if one includes the "divided powers" $(X_i^\pm)^{(k)} = \frac{(X_i^\pm)^k}{[k]_{q_i}!}$ ("restricted specialization") or not ("unrestricted specialization"); the representation theory is quite different in these cases. We will mostly work in the unrestricted specialization, however since we are really only interested in certain representations, it will become clear from the context what is appropriate.

Often the following operators are often more useful:

$$h_i = d_i H_i, \quad e_{\pm i} = \sqrt{[d_i]} X_i^\pm, \quad (1.22)$$

Then the first two relations in (1.15) become

$$[h_i, e_{\pm j}] = \pm(\alpha_i, \alpha_j) e_{\pm j}, \quad (1.23)$$

$$[e_i, e_{-j}] = \delta_{i,j} [h_i]_q. \quad (1.24)$$

In order to have the standard Physics normalization for angular momenta, the normalization of the Killing form will be chosen so that the short roots have length $d_i = \frac{1}{2}$, i.e. $(\alpha_i, \alpha_i) = 1$, and the long roots have length 1. In any case, a rescaling of the Killing form can be absorbed by a redefinition of q .

One could also define another Hopf algebra with reversed coproduct $\Delta'(x) = \tau \circ \Delta(x)$, and S^{-1} instead of S . However this is essentially the same. The reason is that \mathcal{U} has the very important property of being *quasitriangular*, i.e. there exists a universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ with the following properties:

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (1.25)$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \quad (1.26)$$

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \quad (1.27)$$

for any $x \in \mathcal{U}$, where lower indices denote the position of the components of \mathcal{R} in the tensor product algebra $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$: if $\mathcal{R} \equiv a_i \otimes b_i$ (summation is understood), then e.g. $\mathcal{R}_{13} \equiv a_i \otimes 1 \otimes b_i$. By considering $(\Delta' \otimes \text{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13}$, one obtains the Quantum Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (1.28)$$

Furthermore, the following properties are a consequence of (1.25) to (1.27):

$$(S \otimes \text{id})\mathcal{R} = \mathcal{R}^{-1}, \quad (1.29)$$

$$(\text{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R}, \quad (1.30)$$

$$(\epsilon \otimes \text{id})\mathcal{R} = (\text{id} \otimes \epsilon)\mathcal{R} = 1. \quad (1.31)$$

The construction of \mathcal{R} and the proof of the relations (1.25) to (1.27) is based on the so-called quantum double construction due to Drinfeld [15]. It turns out that the Borel subalgebras \mathcal{B}^- and \mathcal{B}^+ , generated by $\{H_i, X_j^-\}$ and $\{H_i, X_j^+\}$ respectively, are Hopf-subalgebras which are dually paired (see below). If $\{a_i\}$ is a basis of \mathcal{B}^- and $\{b_i\}$ the dual basis of \mathcal{B}^+ , then $\mathcal{R} = a_i \otimes b_i$, after factoring out a copy of the Cartan subalgebra which has been counted twice. The relations (1.25) to (1.27) are then easy to see.

This universal \mathcal{R} is the essential feature of a quantum group, and we will make extensive use of it. It incorporates the additional structure of Lie groups which is not used in the classical theory, namely the existence of a certain Poisson structure compatible with the group structure. Furthermore, all this is combined into objects which are holomorphic in q . Therefore one should expect that there is a lot to say about this rich structure.

1.1.3 $\text{Fun}(G_q)$ and dually paired Hopf Algebras

Before studying \mathcal{U} any further, let us now sketch the second approach to quantum groups; for a general review see [6]. It is based on the observation that any (compact) Lie group G with Lie algebra \mathfrak{g} is actually a (coboundary) Poisson–Lie group, i.e. there is a particular Poisson structure on the group manifold which can be written in terms of a "classical r-matrix" $r \in \mathfrak{g} \otimes \mathfrak{g}$, and enjoys certain compatibility conditions. r can again be obtained from a "double construction" [15]; it is *not* given by the structure constants of G , it is truly an *additional* structure on G . Now as in Quantum Mechanics on a phase space, this Poisson structure can be quantized, giving rise to a

non-commutative algebra $Fun(G_q)$ which replaces the commutative algebra $Fun(G)$. If one writes $q = e^h$, h plays the role of the Planck constant, and the classical case $Fun(G)$ is obtained in the limit $h \rightarrow 0$, i.e. $q \rightarrow 1$. This is the origin of the name "Quantum group", and even though this quantization procedure may be formal, the final result is known to exist. Upon this quantization, the "classical r-matrix" r turns into the universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$.

These two approaches are in fact dual to each other. This means that $\mathcal{U}_{q^{-1}} \equiv U_{q^{-1}}(g)$ and $\mathcal{A} = Fun(G_q)$ are dually paired Hopf algebras (notice the replacement $q \rightarrow q^{-1}$). In general, two Hopf algebras \mathcal{U} and \mathcal{A} are said to be dually paired if there exists a non-degenerate inner product $\langle , \rangle : \mathcal{U} \otimes \mathcal{A} \rightarrow \mathbb{C}$, such that:

$$\begin{aligned} \langle xy, a \rangle &= \langle x \otimes y, \Delta(a) \rangle \equiv \langle x, a_{(1)} \rangle \langle y, a_{(2)} \rangle, \\ \langle x, ab \rangle &= \langle \Delta(x), a \otimes b \rangle, \\ \langle S(x), a \rangle &= \langle x, S(a) \rangle, \\ \langle x, 1 \rangle &= \epsilon(x), \quad \text{and} \quad \langle 1, a \rangle = \epsilon(a), \end{aligned} \tag{1.32}$$

for all $x, y \in \mathcal{U}$ and $a, b \in \mathcal{A}$.

The algebra $Fun(G_q)$ can be written down explicitly [18] if it is written as a pseudo matrix group [61], generated by the elements of a $N \times N$ matrix $A \equiv (A^i_j)_{i,j=1\dots N} \in M_N(Fun(G_q))^2$. The coproduct on $Fun(G_q)$ is defined as classically,

$$\Delta A = A \dot{\otimes} A, \quad \text{i.e.} \quad \Delta(A^i_j) = A^i_k \otimes A^k_j, \tag{1.33}$$

and $S(A^i_j) = (A^{-1})^i_j$, $\epsilon(A^i_j) = \delta^i_j$. Now if \langle , \rangle is a dual pairing of $\mathcal{U}_{q^{-1}}$ with $Fun(G_q)$, then this implies that $\pi^i_j \equiv \langle \cdot, A^i_j \rangle$ is a representation of $\mathcal{U}_{q^{-1}}$, i.e.

$$\begin{aligned} \pi^i_j : \mathcal{U}_{q^{-1}} &\rightarrow \mathbb{C}, \\ \pi^i_j(xy) &= \sum_k \pi^i_k(x) \pi^k_j(y), \quad \forall x, y \in \mathcal{U}_{q^{-1}}; \end{aligned} \tag{1.34}$$

we will say much more about representations in a later section. In this representation, the universal $\mathcal{R} \in \mathcal{U}_{q^{-1}} \otimes \mathcal{U}_{q^{-1}}$ gives the numerical R -matrix:

$$\langle \mathcal{R}, A^i_k \otimes A^j_l \rangle = R^{ij}_{kl}. \tag{1.35}$$

Now the definition of a dual pairing (1.32) and (1.27) imply [15, 18]

$$\langle x, A^j_s A^i_r \rangle = \langle \Delta x, A^j_s \otimes A^i_r \rangle$$

²This corresponds to $GL_q(N)$ unless there are explicit or implicit restrictions on the matrix elements of A .

$$\begin{aligned}
&= \langle \tau \circ \Delta x, A^i_r \otimes A^j_s \rangle \\
&= \langle \mathcal{R}(\Delta x) \mathcal{R}^{-1}, A^i_r \otimes A^j_s \rangle \\
&= \langle x, R^{ij}_{kl} A^k_m A^l_n (R^{-1})^{mn}_{rs} \rangle,
\end{aligned} \tag{1.36}$$

for any $x \in \mathcal{U}_{q^{-1}}$, i.e. the matrix elements of A satisfy the commutation relations

$$R^{ij}_{kl} A^k_m A^l_n = A^j_s A^i_r R^{rs}_{mn}, \tag{1.37}$$

which can be written more compactly in tensor product notation as follows:

$$R_{12} A_1 A_2 = A_2 A_1 R_{12}; \tag{1.38}$$

$$R_{12} = (\pi_1 \otimes \pi_2)(\mathcal{R}) \equiv \langle \mathcal{R}, A_1 \otimes A_2 \rangle. \tag{1.39}$$

Starting from this formalism, one can introduce differential forms etc. and study the noncommutative differential geometry of quantum groups, see [62, 64, 51, 57]. So far, we are considering all algebras over C without any reality structure, which we will discuss below.

Now one can recast the commutation relations of $U_{q^{-1}}(g)$ into a more compact form [18]. For a representation π , define matrices

$$\begin{aligned}
L^+_\pi &\equiv (\text{id} \otimes \pi)(\mathcal{R}), \\
SL^-_\pi &\equiv (\pi \otimes \text{id})(\mathcal{R}), \\
L^-_\pi &\equiv (\pi \otimes \text{id})(\mathcal{R}^{-1}).
\end{aligned} \tag{1.40}$$

Then the commutation relations for these matrices follow from the quantum Yang-Baxter equation, e.g.

$$0 = (\text{id} \otimes \pi \otimes \pi)(\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} - \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23}) \tag{1.41}$$

$$= R_{12} L^+_2 L^+_1 - L^+_1 L^+_2 R_{12} \tag{1.42}$$

and similarly

$$R_{12} L^-_2 L^-_1 = L^-_1 L^-_2 R_{12}, \tag{1.43}$$

$$R_{12} L^+_2 L^-_1 = L^-_1 L^+_2 R_{12}. \tag{1.44}$$

The coproduct is now

$$\Delta L^\pm = L^\pm \dot{\otimes} L^\pm, \tag{1.45}$$

and $\epsilon(L^\pm) = I$, $S(L^\pm) = (L^\pm)^{-1}$. The X_i^\pm can be extracted from the upper resp. lower triangular matrices L^\pm [18].

One can also turn the logic around and show that there exists a dual pairing between $U_{q^{-1}}(g)$ and the Hopf algebra defined by (1.38) and (1.33) (with some suitable additional constraints depending on the group, cp. [41]), which can be seen to be a quantization of $Fun(G)$.

1.1.4 More Properties of $U_q(g)$

Let us describe \mathcal{U} in more detail. In the classical case, the Weyl group \mathcal{W} acting on weight space by the reflections σ_i along the simple roots can be "lifted" to an action of the braid group³ with generators T_i on representations of g , in particular on g itself with the adjoint representation. The relations of the braid group are $(T_i T_j)^{m_{ij}} = 1$ if $(\sigma_i \sigma_j)^{m_{ij}} = 1$, but the square of T_i is not required to be 1 any more. The same can be done for \mathcal{U} [37]: there exist algebra automorphisms of \mathcal{U} defined as

$$\begin{aligned} T_i(H_j) &= H_j - A_{ij}H_i, \quad T_i X_i^+ = -X_i^- q_i^{H_i}, \\ T_i(X_j^+) &= \sum_{r=0}^{-A_{ji}} (-1)^{r-A_{ji}} q_i^{-r} (X_i^+)^{(-A_{ji}-r)} X_j^+ (X_i^+)^{(r)} \end{aligned} \quad (1.46)$$

where $(X_i^\pm)^{(k)} = \frac{(X_i^\pm)^k}{[k]_{q_i}!}$, and similarly for lowering operators⁴. If $\omega = \sigma_{i_1} \dots \sigma_{i_N}$ is a reduced expression for the longest element of the Weyl group, then $\{\beta_1 = \alpha_{i_1}, \beta_2 = \sigma_{i_1}(\alpha_{i_2}), \dots, \beta_N = \sigma_{i_1} \dots \sigma_{i_{N-1}}(\alpha_{i_N})\}$ is an ordered set of positive roots. Now one can define root vectors of \mathcal{U} as $X_{\beta_r}^\pm = T_{i_1} \dots T_{i_{r-1}}(X_{i_r}^\pm)$. This can be used to obtain a Poincaré–Birkhoff–Witt basis of $\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+$ where \mathcal{U}^\pm is generated by the X_i^\pm and \mathcal{U}^0 by the H_i : for $\underline{k} = (k_1, \dots, k_N)$, let $X_{\underline{k}}^+ = (X_{\beta_N}^+)^{k_N} \dots (X_{\beta_1}^+)^{k_1}$ and similarly for $X_{\underline{k}}^-$. Then the $X_{\underline{k}}^\pm$ form a P.B.W. basis of \mathcal{U}^\pm , and similarly for \mathcal{U}^- [38].

Using this, one can find explicit formulas for $\mathcal{R} = \mathcal{R}(q)$ [32, 33]. They are somewhat complicated however, and all we need for now is the following form:

$$\mathcal{R} = q^{\sum (\alpha^{-1})_{ij} h_i \otimes h_j} \left(\sum c_{k,k'}(q) \tilde{X}_{\underline{k}}^+ \otimes \tilde{X}_{\underline{k}'}^- \right) \quad (1.47)$$

where⁵ $(\alpha)_{ij} = (\alpha_i, \alpha_j)$, $\tilde{X}_{\underline{k}}^\pm$ is defined like $X_{\underline{k}}^\pm$ with X_i^\pm replaced by $\tilde{X}_i^\pm \equiv q_i^{\pm \frac{1}{2} H_i} X_i^\pm$, and $c_{k,k'}(q) \in \mathbb{C}$ are rational functions of q . Furthermore, the coefficients in (1.47) are uniquely determined by the properties (1.25) to (1.27) [16, 32]. Using this, is easy to

³this has nothing to do with the representations of the braid group obtained from \mathcal{R} .

⁴the T_i can actually be implemented as $\omega_i(\dots)\omega_i^{-1}$ in an extension of \mathcal{U} , see [33, 34]; we will come back to that.

⁵Since we will work with "nice" representations, \mathcal{R} converges.

see that

$$\mathcal{R}(q^{-1}) = \mathcal{R}^{-1}(q), \quad (1.48)$$

$$(\theta \otimes \theta)\mathcal{R}_{12} = \mathcal{R}_{21}. \quad (1.49)$$

For the case of $U_q(sl(2))$, the explicit form is

$$\mathcal{R}(q) = q^{\frac{1}{2}H \otimes H} \left(\sum_{l \geq 0} q^{\frac{1}{2}l(l-1)} \frac{(1 - q^{-2})^l}{[l]_q!} (q^{\frac{1}{2}H} X^+)^l \otimes (q^{-\frac{1}{2}H} X^-)^l \right) \quad (1.50)$$

where we have set $d = (\alpha, \alpha)/2 = 1$, i.e. $q_i = q$.

It is easy to check that the square of the antipode is an inner automorphism

$$S^2(x) = q^{2\tilde{\rho}} x q^{-2\tilde{\rho}}, \quad (1.51)$$

where $\tilde{\rho} = \sum_{\alpha > 0} h_\alpha$ and $h_\alpha = \sum n_i h_i$ if $\alpha = \sum n_i \alpha_i$ [15]. As shown in [15], there is another element in \mathcal{U} with that property, namely $u \equiv S\mathcal{R}_2\mathcal{R}_1$. Therefore

$$v \equiv (S\mathcal{R}_2)\mathcal{R}_1 q^{-2\tilde{\rho}} \quad (1.52)$$

commutes with any element in \mathcal{U} and will be called Drinfeld–Casimir. It satisfies

$$\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1} v \otimes v, \quad (1.53)$$

$$v^{-1} = q^{2\tilde{\rho}} \mathcal{R}_2 S^2(\mathcal{R}_1), \quad (1.54)$$

$$S(v) = v \quad (1.55)$$

where $\mathcal{R}_{12} = \mathcal{R}$ and $\mathcal{R}_{21} = \tau \circ \mathcal{R}$. Furthermore, it is easy to check from (1.49) that

$$\theta(v) = v. \quad (1.56)$$

The value of v on a highest weight representation was first determined in [49], and can be obtained easily from (1.47): if w_λ is a h.w. vector with $X^+ \cdot w_\lambda = 0$ and $h_i \cdot w_\lambda = (\lambda, \alpha_i) w_\lambda$ (see section 1.2), then

$$v \cdot w_\lambda = q^{-c_\lambda} w_\lambda, \quad (1.57)$$

where $c_\lambda = (\lambda, \lambda + 2\rho)$ is the value of the *classical* quadratic Casimir on w_λ .

Later we will need analogs of \mathcal{R} for $\mathcal{U}^{\otimes l}$ with $l > 2$. If $\Delta_{(l)}(x) \in \mathcal{U}^{\otimes l}$ is the (unique) l -fold coproduct of $x \in \mathcal{U}$, let

$$\mathcal{R}_{12\dots l}^{(a)} \equiv (\mathcal{R}_{12\dots(l-1)}^{(a)} \otimes 1)(\Delta_{(l-1)} \otimes \text{id})\mathcal{R}_{12}, \quad (1.58)$$

$$\mathcal{R}_{12\dots l}^{(b)} \equiv (1 \otimes \mathcal{R}_{12\dots(l-1)}^{(b)})(\text{id} \otimes \Delta_{(l-1)})\mathcal{R}_{12} \quad (1.59)$$

for $l > 2$. They have similar properties as \mathcal{R} :

Lemma 1.1.1

$$\mathcal{R}_{12\dots l}^{(a)} = \mathcal{R}_{12\dots l}^{(b)} \equiv \mathcal{R}_{12\dots l}, \quad (1.60)$$

and

$$\Delta'_{(l)}(x) = \mathcal{R}_{12\dots l} \Delta_{(l)}(x) \mathcal{R}_{12\dots l}^{-1}, \quad (1.61)$$

$$\mathcal{R}_{12\dots l}(q^{-1}) = \mathcal{R}_{12\dots l}^{-1}(q), \quad (1.62)$$

$$(\theta \otimes \dots \otimes \theta) \mathcal{R}_{12\dots l} = \mathcal{R}_{l\dots 21}. \quad (1.63)$$

Proof (1.60) follows by straightforward induction: For $l = 3$, it reduces to the Yang–Baxter equation. By the induction assumption, we have

$$\begin{aligned} \mathcal{R}_{12\dots l}^{(a)} &= (\mathcal{R}_{12\dots(l-1)} \otimes 1)(\text{id} \otimes \Delta_{(l-2)} \otimes \text{id})(\Delta \otimes 1) \mathcal{R}_{12} \\ &= (\mathcal{R}_{12\dots(l-1)}^{(b)} \otimes 1)(\text{id} \otimes \Delta_{(l-2)} \otimes \text{id}) \mathcal{R}_{13} \mathcal{R}_{23} \\ &= (1 \otimes \mathcal{R}_{12\dots(l-2)} \otimes 1)(\text{id} \otimes \Delta_{(l-2)} \otimes \text{id}) \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23}. \end{aligned} \quad (1.64)$$

Similarly,

$$\begin{aligned} \mathcal{R}_{12\dots l}^{(b)} &= (1 \otimes \mathcal{R}_{12\dots(l-1)}^{(a)})(\text{id} \otimes \Delta_{(l-2)} \otimes \text{id})(\text{id} \otimes \Delta) \mathcal{R}_{12} \\ &= (1 \otimes \mathcal{R}_{12\dots(l-2)} \otimes 1)(\text{id} \otimes \Delta_{(l-2)} \otimes \text{id}) \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}, \end{aligned} \quad (1.65)$$

and (1.60) follows using the Yang–Baxter equation. Applying $(\Delta_{(l-1)} \otimes \text{id})$ to (1.27), one obtains $(\Delta_{(l-1)} x_{(2)}) \otimes x_{(1)} = ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{12}) \Delta_{(l)}(x) ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{12}^{-1})$, and by induction and using (1.60) it follows

$$(\Delta'_{(l-1)} x_{(2)}) \otimes x_{(1)} = (\mathcal{R}_{12\dots(l-1)} \otimes 1)((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{12}) \Delta_{(l)}(x) \quad (1.66)$$

$$\begin{aligned} &\cdot ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{12}^{-1}) (\mathcal{R}_{12\dots(l-1)}^{-1} \otimes 1) \\ &= \mathcal{R}_{12\dots l}^{(a)} \Delta_{(l)}(x) (\mathcal{R}_{12\dots l}^{(a)})^{-1}, \end{aligned} \quad (1.67)$$

which shows (1.61).

For illustration let us also show (1.63). From (1.58), one gets by induction

$$\begin{aligned} (\theta \otimes \dots \otimes \theta) \mathcal{R}_{12\dots l}^{(a)} &= ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{21}) (\mathcal{R}_{(l-1)\dots 21} \otimes 1) \\ &= (\mathcal{R}_{(l-1)\dots 21} \otimes 1) ((\Delta'_{(l-1)} \otimes \text{id}) \mathcal{R}_{21}) \\ &= \mathcal{R}_{l\dots 21}^{(b)} = \mathcal{R}_{l\dots 21}, \end{aligned} \quad (1.68)$$

using the flipped (1.61) and (1.60). Similarly one can see (1.62). \square

All this will become more obvious in section 4.3.

1.1.5 Reality Structures

So far we considered all algebras over \mathcal{C} , in particular \mathcal{U} is the universal enveloping algebra of a complex Lie algebra. If one wants to consider e.g. $SO_q(5) \equiv U_q(so(5; \mathbb{R}))$ or $SO_q(2, 3) \equiv U_q(so(2, 3; \mathbb{R}))$, one has to introduce a star-structure, an *antilinear* involution \bar{x} , as classically. This star-structure must be such that if $x \in \mathcal{U}$ is a group element for $q = 1$, e.g. $x = \exp(y)$ for y an element in the Lie algebra, then $\bar{x} = x^{-1}$, i.e. \bar{x} becomes the adjoint of x in a unitary representation of \mathcal{U} (this will be considered in detail below).

There are several possible star-structures on \mathcal{U} . One has to distinguish the cases $q \in \mathbb{R}$ and $|q| = 1$. If $q \in \mathbb{R}$, then a natural definition is

$$\bar{x}^r \equiv \theta(x^*) \quad (1.69)$$

$$\overline{x \otimes y}^r \equiv \bar{x}^r \otimes \bar{y}^r, \quad (1.70)$$

with

$$\overline{\Delta(x)}^r = \Delta(\bar{x}^r) \quad (1.71)$$

and $\overline{Sx}^r = S^{-1}\bar{x}^r$, which is a standard Hopf algebra star-structure (x^* is the complex conjugate of $x \in \mathcal{U}$, where the X_i^\pm are considered real). Using the uniqueness of \mathcal{R} (see the discussion below (1.47)), one can see that

$$\overline{\mathcal{R}_{12}}^r = \mathcal{R}_{21} = \tau \circ \mathcal{R}. \quad (1.72)$$

If $|q| = 1$, a natural definition is

$$\bar{x}^c \equiv \theta(x^*) \quad (1.73)$$

$$\overline{x \otimes y}^c \equiv \bar{y}^c \otimes \bar{x}^c \quad (1.74)$$

with

$$\overline{\Delta(x)}^c = \Delta(\bar{x}^c) \quad (1.75)$$

and $\overline{Sx}^c = S\bar{x}^c$, which is a *nonstandard* Hopf algebra star-structure; notice that $\bar{q} = q^{-1}$. In this case, $\overline{\mathcal{R}}^c = \mathcal{R}^{-1}$, and more generally $\overline{\mathcal{R}_{12\dots l}^{(a)}}^c = (\mathcal{R}_{12\dots l}^{(b)})^{-1}$ with the obvious extension of (1.74) to several factors, thus

$$\overline{\mathcal{R}_{12\dots l}}^c = \mathcal{R}_{12\dots l}^{-1} \quad (1.76)$$

using Lemma 1.1.1.

Furthermore, $\overline{(S\mathcal{R}_2)\mathcal{R}_1}^c = \mathcal{R}_2^{-1}S\mathcal{R}_1^{-1} = \mathcal{R}_2S^2\mathcal{R}_1 = (S\mathcal{R}_2\mathcal{R}_1)^{-1}$ (see [6]), so

$$\overline{v}^c = v^{-1}. \quad (1.77)$$

Both \overline{x}^r and \overline{x}^c correspond to the *compact* case, such as $SO_q(5)$; however we will mainly be interested in the case of $|q| = 1$. Having applications in QFT in mind, one might then be worried about (1.74). We will see in sections 4.4, 4.2 and to some extent in section 1.2.1 how this can be used consistently with a many-particle interpretation. In the classical case, the coproduct on \mathcal{U} is cocommutative, and it does not make any difference whether its components are flipped by the reality structure or not.

Reality structures corresponding to noncompact groups can be obtained from \overline{x}^c by conjugation with elements of the Cartan subalgebra. We will only consider star-structures of the form $\overline{X_i^+} = \pm X_i^-$ and $\overline{H_i} = H_i$. For example, $SO_q(2, 1)$ is the algebra $U_q(sl(2, C))$ with $\overline{H} = H$ and $\overline{X^\pm} = -X^\mp$, which can be realized as $\overline{x} = (-1)^{-H/2}\overline{x}^c(-1)^{H/2}$. Then for $|q| = 1$, again $\overline{\mathcal{R}} = ((-1)^{-H/2} \otimes (-1)^{-H/2})\mathcal{R}^{-1}$. $((-1)^{H/2} \otimes (-1)^{H/2}) = \mathcal{R}^{-1}$. The cases $SO_q(2, 1)$ and $SO_q(2, 3)$ will be considered in much more detail below. We will only consider star-structures with

$$\overline{\mathcal{R}} = \mathcal{R}^{-1} \quad (1.78)$$

for $|q| = 1$.

1.2 Representation Theory

We will only consider the representation theory of \mathcal{U} . The main advantage of this point of view is that the representation theory of \mathcal{U} can be formulated in the familiar language of ordinary semi-simple Lie algebras.

Let us first collect the basic definitions. We will (essentially) only consider finite-dimensional representations. A representation of \mathcal{U} on a vectorspace V is a map $\mathcal{U} \rightarrow GL(V)$ such that $(xy) \cdot v = x \cdot (y \cdot v)$ and $1 \cdot v = v$ (sometimes we will omit the \cdot). Then one can as usual diagonalize the Cartan subalgebra, and every representation is a sum of weight spaces. A vector v_λ has weight λ if $h_i v_\lambda = (\lambda, \alpha_i) v_\lambda$. Then X_i^\pm are rising and lowering operators as in the classical case, since (1.12) and (1.23) are undeformed. We will mainly (but not always) consider the case of *integral* weights, i.e. $(\lambda, \beta_i) \in d_i \mathbb{Z}$. Classically, all irreducible representations are highest weight representations V with *dominant* integral highest weight λ , i.e. $V = \mathcal{U}w_\lambda$ and

$(\lambda, \alpha_i) > 0$ for all simple roots α_i . Finally, let $Q = \sum \mathbb{Z}\alpha_i$ be the root lattice and $Q^+ = \sum \mathbb{Z}_+\alpha_i$ where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. We will write

$$\lambda \succ \mu \quad \text{if} \quad \lambda - \mu \in Q^+. \quad (1.79)$$

Given two representations V_1, V_2 of a Hopf algebra, the tensor product representation is naturally defined as $x \cdot (v_1 \otimes v_2) \equiv \Delta(x)(v_1 \otimes v_2) = (x_{(1)} \cdot v_1) \otimes (x_{(2)} \cdot v_2)$.

So far, we have not specified q at all. For the representation theory however, there are two very different cases. One is if q is "generic", i.e. *not* a root of unity, and the other is if q is a root of unity.

If q is generic, then the representation theory is essentially the same as in the classical case, in the sense that all the important theorems have a perfect analog. This is quite intuitive, since everything will be holomorphic in q , which as always is a very strong constraint. We will quote the main results in a moment. If q is a root of unity however, the representation theory changes completely, and essentially none of the classical theorems continue to hold. Roughly speaking this happens because poles resp. zeros occur in various quantities. While it is more complicated and therefore often discarded, this is the truly interesting case. The main objective of this thesis is to point out that many of its features seem to be very relevant to Quantum Field Theory, and not only to Conformal Field Theory. In any case, the root of unity case is not well enough understood, and deserves further study.

Consider first the case of generic q . Then the basic results are as follows:

- Any finite-dimensional representation of \mathcal{U} is completely reducible, i.e. it decomposes into a direct sum of irreducible representations (irreps).
- The irreps are highest weight representations with dominant integral highest weight λ , and the representation space is the same as classically.
- The fusion rules are the same as classically.

So the Weyl group acts on the weights of an irrep, and in fact a braid group action can be defined on any representation [39, 25, 33]. Complete reducibility was first proved by Rosso [50].

These results are not hard to understand. The first two would be obvious if one could apply the fact that as *algebra*, $\mathcal{U} = U_q(g)$ is nothing but the classical enveloping algebra [17] (with a *formal* variable q however, and the correspondence may not be realized for a given $q \in \mathbb{C}$). Since similar issues will arise later, we want to explain here

why a representation V of \mathcal{U} which is irreducible for $q = 1$ can become reducible only for q a root of unity. Such a representation must have a dominant integral highest weight λ . If it contains a submodule at q_0 with a highest weight vector with weight λ_0 (which is dominant again by virtue of the Weyl group resp. braid group action), then the Drinfeld Casimir v must be the same on this submodule, so $q_0^{c_\lambda} = q_0^{c_{\lambda_0}}$ using (1.57). However $\lambda \succ \lambda_0$, and we have

Lemma 1.2.1 *If λ, λ_0 are dominant weights with $\lambda \succ \lambda_0$, then*

$$c_\lambda > c_{\lambda_0}. \quad (1.80)$$

Proof Notice first that $c_\lambda \equiv (\lambda, \lambda + 2\rho) = c_{\tilde{\sigma}_i(\lambda)}$, where $\tilde{\sigma}_i(\lambda) = \lambda - \frac{(\lambda + \rho, \beta_i)}{d_i} \beta_i$ is the modified action of the Weyl reflection along any root β_i with reflection center $-\rho$. Since $\lambda \succ \lambda_0$ and both are dominant, λ_0 is contained in the convex hull of λ and the $\tilde{\sigma}_i(\lambda)$. But the Killing form is Euclidean and therefore convex, and the statement follows. \square

Therefore if this V is not irreducible at q_0 , q_0 must be a phase, and in fact a root of unity (we assume $q_0 \neq 0$).

Complete reducibility can be understood using the concept of invariant sesquilinear forms.

1.2.1 Invariant Forms, Verma Modules and Unitary Representations

A bilinear form $(\ , \)$ on a representation V is linear in both arguments, while a sesquilinear form is linear in the second argument and antilinear in the first.

A bilinear form is called *invariant* if

$$(u, x \cdot v) = (\theta(x) \cdot u, v) \quad (1.81)$$

for $u, v \in V$; this is sometimes called covariant [13]. This can be considered for any $q \in \mathbb{C}$.

For $q \in \mathbb{R}$ or $|q| = 1$, consider a star-structure as in section 1.1.5 and denote it by \overline{x} , so $\overline{X_i^\pm} = \pm X_i^\mp$ and $\overline{H_i} = H_i$. Then a sesquilinear form $(\ , \)$ is called *invariant* if

$$(u, x \cdot v) = (\overline{x} \cdot u, v) \quad (1.82)$$

for $u, v \in V$; this is also sometimes called covariant. It is *hermitian* if

$$(u, v)^* = (v, u). \quad (1.83)$$

A hermitian sesquilinear form is called an *inner product*. Note that we always consider q to be a complex number, so $\bar{q} = q^*$; in the literature, q is often treated as an indeterminate, and our definitions agree with those of e.g. [13] only for $|q| = 1$, which is the case we are mainly interested in. Finally, a representation V is *unitary* or *unitarizable* if there exists a positive definite invariant inner product on V .

Given a highest weight (h.w.) representation $V(\lambda)$ with h.w. vector w_λ , there is a unique invariant inner product $(\ , \)$ on $V(\lambda)$ for $q \in \mathbb{R}$ or $|q| = 1$, resp. an invariant symmetric bilinear form for any $q \in \mathbb{C}$. Uniqueness is clear, since one can express any $(\mathcal{U} \cdot w_\lambda, \mathcal{U} \cdot w_\lambda)$ in terms of $(w_\lambda, \mathcal{U}^0 \cdot w_\lambda)$ or $(\mathcal{U}^0 \cdot w_\lambda, w_\lambda)$ using invariance and the commutation relations. These two results agree and $(\ , \)$ is hermitian, because the star-structure is consistent with the commutation relations; notice that $[h_i]_q \in \mathbb{R}$.

Thinking of applications in Quantum Mechanics, the importance of unitarity is obvious. But invariant sesquilinear forms (or bilinearforms) are also very useful as technical tools, due to the following well-known observation: if a highest weight representation $V(\lambda)$ is not irreducible, it contains a submodule. Now all these submodules are null spaces w.r.t. the sesquilinear form, i.e. they are orthogonal to any state in $V(\lambda)$. Therefore one can consistently factor them out, and obtain a sesquilinear form on the quotient space. To see that they are null, let $v_\mu \in V(\lambda)$ be in some submodule, i.e. $w_\lambda \notin \mathcal{U} \cdot v_\mu$. Now for any $v \in \mathcal{U} \cdot w_\lambda$, it follows $(v_\mu, v) \in (\mathcal{U}v_\mu, w_\lambda) = 0$, using invariance and the fact that there is only one vector with weight λ in the h.w. representation $V(\lambda)$, namely w_λ . Conversely,

Lemma 1.2.2 *Let w_λ be the highest weight vector of an irreducible highest weight representation $L(\lambda)$ with invariant inner product. If $(w_\lambda, w_\lambda) \neq 0$, then $(\ , \)$ is nondegenerate, i.e.*

$$\det(L(\lambda)_\eta) \neq 0 \quad (1.84)$$

for every weight space with weight $\lambda - \eta$ in $L(\lambda)$.

Proof Assume to the contrary that there is a vector v_μ which is orthogonal to all vectors of the same weight, and therefore to all vectors of any weight. Because $L(\lambda)$ is irreducible, there exists an $u \in \mathcal{U}$ such that $w_\lambda = u \cdot v_\mu$. But then $(w_\lambda, w_\lambda) = (w_\lambda, u \cdot v_\mu) = (\bar{u} \cdot w_\lambda, v_\mu) = 0$, which is a contradiction. \square

Verma modules For any weight λ , there exists a "universal" highest weight module, the *Verma module* $M(\lambda)$. It is the (unique) \mathcal{U} - module having a h.w. vector w_λ such that the vectors $X_{\underline{k}}^- w_\lambda$ form a basis of $M(\lambda)$ [13], where the $X_{\underline{k}}^-$ are a P.B.W. basis of \mathcal{U}^- . This is the only infinite-dimensional representation we will consider, and only as a technical tool. The importance of Verma modules lies in the fact that all highest weight representations can be obtained from $M(\lambda)$ by factoring out an appropriate submodule. In particular, the (unique) irrep $L(\lambda)$ with h.w. λ is obtained from $M(\lambda)$ by factoring out its maximal proper submodule. Since it is a highest weight module, one can define a unique invariant inner product $(\ , \)$ on a Verma module for $|q| = 1$ and $q \in \mathbb{R}$, and its maximal proper submodule is precisely the corresponding null subspace (see [13] on how to define analogous forms for generic q).

Forms on tensor products Now let V_i be h.w. representations of \mathcal{U} for any $q \in \mathbb{C}$ with dominant integral highest weight μ_i , such that the V_i are irreducible as long as q is not a root of unity. Therefore on each V_i there is an invariant inner product $(\ , \)_i$ for $q \in \mathbb{R}$ and for $|q| = 1$, which is non-degenerate if q is not a root of unity. It is important to realize that the representation *space* V_i is the same for all q (in particular for $q = 1$), only the action of \mathcal{U} on it depends on q , and is in fact analytic (one way to see this is to use a P.B.W. basis, another is to construct the V_i by taking suitable tensor products, as we will see in a moment). Let

$$V \equiv V_1 \otimes \dots \otimes V_r, \quad (1.85)$$

and for $a = a_1 \otimes \dots \otimes a_r \in V_1 \otimes \dots \otimes V_r$ and $b \in V$, define

$$(a, b)_\otimes \equiv (a_1, b_1)_1 \dots (a_r, b_r)_r. \quad (1.86)$$

We claim that for $q \in \mathbb{R}$, $(\ , \)_\otimes$ is a positive-definite inner product:

$(\ , \)_\otimes$ is invariant because of (1.70) for $q \in \mathbb{R}$, and it is certainly hermitian and positive definite if the $(\ , \)_i$ are. Let $M_{k_i, l_i}^{(i)}$ be the hermitian matrix of $(\ , \)_i$ in some basis of V_i . Since the $(\ , \)_i$ are determined by the algebra alone, the $M_{k_i, l_i}^{(i)}$ are certainly continuous (and can be extended to analytic objects), so their eigenvalues are real and continuous. Since V_i remains irreducible for $q \in \mathbb{R}$ as shown above and the eigenvalues are positive for $q = 1$, they cannot vanish for $q \in \mathbb{R}$ because of Lemma 1.2.2. So $(\ , \)_\otimes$ is indeed a positive-definite invariant inner product. Similarly, $(\ , \)_\otimes$ is an invariant bilinear form for any $q \in \mathbb{C}$ if it is built from bilinear forms on the V_i .

Now for $q \in \mathbb{R}$, one can use the Gram-Schmidt orthogonalization method as usual, and V is the direct sum of orthogonal highest weight irreps V_{λ_i} with the same

highest weights λ_l and multiplicities m_{λ_l} as classically. This implies that for $q \in \mathbb{R}$, the Drinfeld Casimir v satisfies the characteristic equation

$$\prod_{\lambda_l} (v - q^{-c_{\lambda_l}}) = 0, \quad (1.87)$$

where the product is over all different highest weights counted *once*, and not with multiplicity m_{λ_l} . Since v is analytic, (1.87) holds for all $q \in \mathbb{C}$, and one can write down the projectors on the eigenspaces of v as

$$P_{\lambda_l} = \frac{\prod_{\lambda_{l'} \neq \lambda_l} (v - q^{-c_{\lambda_{l'}}})}{\prod_{\lambda_{l'} \neq \lambda_l} (q^{-c_{\lambda_l}} - q^{-c_{\lambda_{l'}}})}, \quad (1.88)$$

with $\sum P_{\lambda_l} = 1$ and $P_{\lambda_l} P_{\lambda_k} = \delta_{\lambda_l, \lambda_k} P_{\lambda_l}$. Their only singularities are isolated poles in q , and it follows that for generic q , the image of P_{λ_l} consists of m_{λ_l} copies of the highest weight irrep V_{λ_l} with h.w. λ_l . In fact using Lemma 1.2.1, this may break down only at roots of unity. Thus we have shown complete reducibility of V for q not a root of unity.

This argument illustrates the use of inner products. From now on, we will only consider the case $|q| = 1$. Then one can define another invariant sesquilinear form on $V = V_1 \otimes \dots \otimes V_r$, namely

$$(a, b)_{\mathcal{R}} \equiv (a, \mathcal{R}_{12\dots l} b)_{\otimes} \quad (1.89)$$

with $\mathcal{R}_{12\dots l}$ as in Lemma 1.1.1. Indeed using (1.75) and (1.74),

$$\begin{aligned} (\bar{x} \cdot a, b)_{\mathcal{R}} &= (\Delta_{(r)}(\bar{x})(a_1 \otimes \dots \otimes a_r), \mathcal{R}_{1\dots r}(b_1 \otimes \dots \otimes b_r))_{\otimes} \\ &= (a_1 \otimes \dots \otimes a_r, \Delta'_{(r)}(x) \mathcal{R}_{1\dots r}(b_1 \otimes \dots \otimes b_r))_{\otimes} \\ &= (a_1 \otimes \dots \otimes a_r, \mathcal{R}_{1\dots r} \Delta_{(r)}(x)(b_1 \otimes \dots \otimes b_r))_{\otimes} \\ &= (a, x \cdot b)_{\mathcal{R}}, \end{aligned} \quad (1.90)$$

since the $(\ , \)_i$ are invariant w.r.t. \bar{x} , and $\Delta'_{(r)}$ is the flipped r -fold coproduct. While it is not positive definite in general, $(\ , \)_{\mathcal{R}}$ is *nondegenerate* if q is not a root of unity, which will be very important later. To see this, let again $M_{k_i, l_i}^{(i)}$ be the invertible matrix of the inner products $(\ , \)_i$ on the irreps V_i . Then the matrix of $(\ , \)_{\mathcal{R}}$ is $\sum_{k'} M_{k_1, k'_1}^{(1)} \dots M_{k_r, k'_r}^{(r)} \mathcal{R}_{l_1, \dots, l_r}^{k'_1, \dots, k'_r}$, which is invertible, because $\mathcal{R}_{12\dots r}$ is invertible. In fact, $(\ , \)_{\mathcal{R}}$ *remains nondegenerate at roots of unity as long as all the V_i remain irreducible*, since then $\mathcal{R}_{12\dots r}$ exists and is invertible on these representations, as we will see in section 1.2.3.

In the classical limit $q \rightarrow 1$, $(\ , \)_{\mathcal{R}}$ reduces to $(\ , \)_{\otimes}$ since $\mathcal{R} \rightarrow 1 \otimes 1$, however it is not hermitian unless $q = 1$ (remember $|q| = 1$). In chapter 4, we will show how one can define a (hermitian) inner product on a "part" of V using a BRST operator for q a root of unity, and in fact a many-particle *Hilbert space*, with the "correct" classical limit.

Therefore we have to study the root of unity case. But first, we briefly discuss the \hat{R} -matrix:

1.2.2 \hat{R} - Matrix and Centralizer Algebra

For a (finite-dimensional) representation V of \mathcal{U} , consider the n -fold tensor product of V with itself,

$$V^{\otimes n} \equiv V \otimes \dots \otimes V. \quad (1.91)$$

This carries a natural representation of \mathcal{U} using the n -fold coproduct $\Delta_{(n)}$. Classically, the symmetric group (or its group algebra) generated by $\tau_{i,i+1}$ which interchanges the factors in position i and $i+1$ commutes with the action of $U(g)$ on $V^{\otimes n}$. The maximal such algebra commuting with the representation is called the centralizer algebra. In the quantum case, there is an analog of this, namely

$$\hat{R}_{i,i+1} \equiv \text{id} \otimes \dots \otimes (\tau \circ (\pi \otimes \pi) \mathcal{R}) \otimes \dots \otimes \text{id} \quad (1.92)$$

where the nontrivial part is in positions i and $i+1$, and π is the representation on V . Notice that such a definition only makes sense for identical representations. It follows from (1.27) and coassociativity that $\hat{R}_{i,i+1}$ commutes with the action of \mathcal{U} on $V^{\otimes n}$. Therefore representations of \mathcal{U} on this space fall into representations of the centralizer algebra. This is familiar from quantum field theory, where bosons and fermions are totally symmetric resp. antisymmetric representations of the permutation group. The Yang-Baxter equation now becomes

$$\hat{R}_{i,i+1} \hat{R}_{i+1,i+2} \hat{R}_{i,i+1} = \hat{R}_{i+1,i+2} \hat{R}_{i,i+1} \hat{R}_{i+1,i+2} \quad (1.93)$$

Acting on $V \otimes V$, (1.53) becomes $\Delta(v) = (\hat{R})^{-2}(v \otimes v)$. Now v is diagonalizable for generic q with eigenvalues q^{-c_λ} , because of complete reducibility. Therefore $(\hat{R})^2$ is diagonalizable, with nonzero eigenvalues. This implies (e.g. using the Jordan normal form, cp. section 4.1) that \hat{R} is diagonalizable for generic q , with eigenvalues $\pm q^{\frac{1}{2}(c_\lambda - 2c_\mu)}$ where μ is the highest weight of V if V is irreducible. Such a result was first obtained in [49] using a different method.

The centralizer algebra provides a connection between quantum groups and conformal field theory [23, 1]. For small representations, it can be described explicitly, and again the root of unity case is very different from the generic case.

1.2.3 Aspects of Representation Theory at Roots of Unity

Generally speaking, the root of unity case is somewhat more complicated than the case of generic q , but also more interesting and probably more relevant to physics. Unfortunately, the general mathematical literature on this subject is not very accessible to physicist⁶. The rank one case (i.e. $U_q(sl(2))$ and its real structures) however is quite instructive and is discussed in [30, 45]. In this section we will only mention a few important features, and many of the later sections will be devoted to studying certain aspects in more detail. In general, it is probably fair to say that the root of unity case is not well enough understood.

First, the subalgebras of $U_q(g)$ generated by X_i^\pm and H_i are nothing but $U_{q_i}(sl(2))$ algebras (the coalgebra structure is not the same, however), with q_i instead of q . Let

$$q = e^{2\pi i n/m} \quad (1.94)$$

with $\gcd(m, n) = 1$, and let $M = m$ if m is odd, and $M = m/2$ if m is even. Similarly for $q_i = e^{2\pi i d_i n/m}$, let $M_{(i)} = m$ if $d_i = \frac{1}{2}$, and $M_{(i)} = M$ if $d_i = 1$ (recall our normalization conventions in section 1.1.2). Then $M_{(i)}$ is the smallest integer such that

$$[M_{(i)}]_{q_i} = 0. \quad (1.95)$$

Highest weight vectors and irreducible representations The following crucial formula can be checked easily [30]:

$$[X_i^+, (X_i^-)^k] = (X_i^-)^{k-1} [k]_{q_i} [H_i - k + 1]_{q_i}. \quad (1.96)$$

In particular, this shows that $(X_i^-)^{M_{(i)}}$ is *central in \mathcal{U}* , and so is $(X_i^+)^{M_{(i)}}$. Now if w_λ is a highest weight vector, then $(X_i^-)^{M_{(i)}} \cdot w_\lambda$ is either zero or again a highest weight vector. In the latter case, the representation contains an invariant submodule. In particular,

$$(X_i^-)^{M_{(i)}} \cdot w_\lambda = 0 \quad (1.97)$$

⁶[13] is among the more readable sources.

on all irreps with highest weight vector w_λ . Due to the braid group action (1.46) resp. algebra automorphism, similar statements apply to all root vectors $X_{\beta_r}^\pm$, and considering the P.B.W. basis of \mathcal{U} , it follows that all irreducible highest weight representations are finite-dimensional at roots of unity. This is very different from the generic case.

Another important feature is the existence of non-trivial one-dimensional representations at roots of unity, namely w_{λ_0} with weight $\lambda_0 = \sum \frac{m}{2n} k_i \alpha_i$ for integers k_i . It is easy to check from (1.15) that this is indeed a representation of \mathcal{U} . By tensoring any representation with w_{λ_0} , one obtains another representation with identical structure, but all weights shifted by λ_0 .

There exist also "cyclic" representations with $(X_i^-)^{M(i)} = \text{const}$ if q^{H_i} is used instead of H_i , see [6].

Assume $V(\lambda)$ is a highest weight module of \mathcal{U} which is analytic in q (i.e. the vector space $V(\lambda)$ is fixed, but the action of U on it depends analytically on q , such as a Verma module with dominant integral λ). The submodules contained in $V(\lambda)$ for generic q will certainly survive at roots of unity, since a highest weight vector w is characterized by $(\sum_i X_i^+) \cdot w = 0$, which at roots of unity may have more, but not fewer solutions than generically. In fact, highest weight modules typically develop additional h.w. vectors at roots of unity. We can see this in the example of a Verma module of $U_q(sl(2))$:

Let $M(j)$ be the Verma module of $U_q(sl(2))$ with highest weight $\lambda = j$, i.e. $H \cdot w_j = j w_j$ and $X^+ \cdot w_j = 0$. Then $M(j)$ has a basis $\{w_j, (X^-)^k \cdot w_j; \quad k \in \mathbb{N}\}$ with weights $j, j-2, \dots$. For generic q , $M(j)$ contains another highest weight vector only if $j \in \mathbb{N}$, namely with weight $-j-2$; this can be seen from (1.96). However for q a root of unity, $[H-k+1]_q = 0$ if $H-k+1 = M$, and (1.96) implies that there is an additional h.w. vector at weight $j-2k = j-2(j+1-M) = 2M-j-2$ (if this is smaller than j and $j \in \mathbb{Z}$), another one at weight $j-2M$, and so on. In fact, the weights of all the h.w. vectors in $M(j)$ can be obtained from j by the action of the "affine Weyl group" generated by reflections σ_l with reflection centers $lM - \rho = lM - 1$, for any $l \in \mathbb{Z}$. An analogous statement (the "strong linkage principle") holds in the higher rank case as well, and will be discussed in section 3.2.3. This can be used to determinine the structure of the irreps of \mathcal{U} .

In summary, the highest weight irreps at roots of unity are "usually" smaller than the irreps with the same highest weight for generic q , and they are always finite-dimensional.

Tensor products The coproduct determines the representation of \mathcal{U} on a tensor product, and for q as in (1.94), one can easily see using a q -binomial theorem that $\Delta(X_i^\pm)^{M(i)} = (X_i^\pm)^{M(i)} \otimes q_i^{M(i)H_i/2} + q_i^{-M(i)H_i/2} \otimes (X_i^\pm)^{M(i)}$, cp. [45]. Therefore if V_i are highest weight irreps, then $(X_i^\pm)^{M(i)} = 0$ on $V_1 \otimes V_2$, and similarly for any number of factors. In this context, it is useful to consider the various quantities as being analytic in $h \equiv q' - q$, where q is fixed to be (1.94). Then e.g. $[M(i)]_{q'_i}$ has a first-order zero in h . In particular, $(X_i^\pm)^{(M(i))} \equiv \frac{(X_i^\pm)^{M(i)}}{[M(i)]_{q'_i}}$ is well-defined, and the distinction between the unrestricted and restricted specialization mentioned in section 1.1.2 becomes important. We will essentially work in the unrestricted specialization, i.e. $(X_i^\pm)^{(M(i))}$ is *not* considered an element of \mathcal{U} .

Consider the tensor product $V_1 \otimes V_2$ of two representations V_1, V_2 which are irreducible for generic q . It is well known (e.g. [45, 30]) that if the V_i are "large enough", $V_1 \otimes V_2$ does *not* decompose into the direct sum of irreps at roots of unity, but different generic irreps ("would-be irreps") in $V_1 \otimes V_2$ combine into irreducible representations; this will be discussed in detail in later sections. One should notice that this can happen because 1) all Casimirs, including the Drinfeld Casimir v , approach the same value on the "would-be irreps" which recombine as $q' \rightarrow q$, and 2) the larger of the recombining "would-be irreps" develops a h.w. vector, which becomes the h.w. vector of the smaller constituent. In other words, the image of different projectors (1.88) becomes linearly dependent at roots of unity, and they develop poles. Nevertheless,

Lemma 1.2.3 *The image $Im(P_\lambda)$ of P_λ (1.88) for any given λ is analytic even at roots of unity, in the sense that there exists an analytic basis for it. In particular, the dimension is the same as generically.*

Proof One can inductively define an analytic basis of $Im(P_\lambda(q'))$ for q' near the root of unity q as follows: Suppose the $\{v_i(q')\}_{i=1}^d$ are analytic and linearly independent at $q' = q$, and satisfy $P_\lambda(q') \cdot v_i(q') = v_i(q')$. If d is smaller than the generic dimension of $Im(P_\lambda)$, take a vector $v_{d+1} \in Im(P_\lambda(q_0))$ for q_0 near q which is not in the span of the $\{v_i(q')\}_{i=1}^d$ at $q' = q_0$. Define $v_{d+1}(q') \equiv h^k P_\lambda(q') \cdot v_{d+1}$, where $k \in \mathbb{Z}$ is such that $v_{d+1}(q')$ is analytic and non-vanishing at $q' = q$ (this is possible because $P_\lambda(q')$ has only poles). Then $v_{d+1}(q')$ satisfies $P_\lambda(q') \cdot v_{d+1}(q') = v_{d+1}(q')$ for $q' \neq q$, since the P_λ are projectors. Furthermore, $\{v_i(q')\}_{i=1}^{d+1}$ are linearly independent except possibly for isolated values of q' , and if they are linearly dependent at $q' = q$, one can redefine $\tilde{v}_{d+1}(q') = \frac{1}{h^k}(\sum v_{d+1}(q') - a_i v_i(q'))$, so that the new $\{v_i\}_{i=1}^{d+1}$ span the same space at $q' \neq q$, are analytic *and* linearly independent at $q' = q$. This is

always possible, because the determinant defined by the $\{v_i(q')\}_{i=1}^{d+1}$ is analytic, but not identically zero. \square

Notice that it is essential that the P_λ have only poles at $q' = q$, and no essential singularities. Furthermore, if a vector w is not in $\oplus_\lambda \text{Im}(P_\lambda)$ at the root of unity, then it is clear that $P_\lambda(q') \cdot w$ will indeed have a pole at $q' = q$ for some P_λ .

\mathcal{R} at roots of unity Finally we need to know whether \mathcal{R} makes sense at roots of unity. This can be answered using the explicit formulas for \mathcal{R} given in [33, 32, 34], refining (1.47). It turns out that \mathcal{R} is built out of $\tilde{\mathcal{R}}_{\beta_i}$, i.e. universal \mathcal{R} 's of the $U_q(\mathfrak{sl}(2))$ subalgebras corresponding to all roots. Looking at (1.50), the term $\frac{1}{[l]_q!}((X^+)^l \otimes (X^-)^l)$ becomes ill-defined at roots of unity. So strictly speaking \mathcal{R} does not exist as element of $\mathcal{U} \otimes \mathcal{U}$, but its action on representations $V_1 \otimes V_2$ is well-defined at roots of unity *provided* $(X_i^\pm)^{M(i)} \cdot V_i$ vanishes on all representations V_i . In particular, \mathcal{R} is well-defined if all V_i are irreps, and then all the formulas for \mathcal{R} hold by analyticity. The same is true for its many-argument cousin $\mathcal{R}_{12\dots l}$.

It should be obvious by now that we are dealing with a structure which is very different from the usual representation theory of Lie groups and algebras. The most remarkable objects however are the indecomposable representations which have barely been mentioned. They will be studied in later sections. But first, we make a digression and consider quantum spaces.

Chapter 2

Quantum Spaces associated to Quantum Groups

The classical Lie groups $SL(N)$, $SO(N)$ and $Sp(N)$ act naturally on N -dimensional vector spaces \mathcal{M} ("vector representation"), preserving certain objects such as volume-elements or bilinear forms. There exists a perfect analog for quantum groups, introduced by Faddeev, Reshetikhin and Takhtadjan [18]. In the spirit of noncommutative geometry, one does not consider the spaces themselves, but the algebras of functions $Fun(\mathcal{M})$ on them, which upon quantization turn into noncommutative algebras $Fun(\mathcal{M}_q)$. Because it is customary in the literature, we will use the dual formulation of quantum groups in this chapter, namely $Fun(G_q)$ as explained in section 1.1.3.

2.1 Definitions and Examples

2.1.1 Actions and Coactions

So far, with "representation" we always meant a *left action* of \mathcal{U} on a vector space V . In this chapter, we will be more explicit, and instead of writing $x \cdot v$ we will write $x \triangleright v$ for $v \in V$. A *left action* of an algebra \mathcal{A} on a vector space V is defined by

$$(xy) \triangleright v = x \triangleright (y \triangleright v), \quad 1 \triangleright v = v \quad (2.1)$$

for $x \in \mathcal{A}$, and V is called a left \mathcal{U} -module. If the representation space is not only a vector space but also an algebra \mathcal{F} and \mathcal{A} is a Hopf algebra (such as $U_q(g)$), we can in addition ask that this action preserve the algebra structure of \mathcal{F} , i.e.

$x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b)$ and $x \triangleright 1 = 1\epsilon(x)$ for all $a, b \in \mathcal{F}$ and $x \in \mathcal{A}$. \mathcal{F} is then called a left \mathcal{A} -module algebra.

Similarly, a *right action* of \mathcal{A} on a vector space V' is defined by

$$v' \triangleleft (xy) = (v' \triangleleft x) \triangleleft y, \quad v' \triangleleft q = v', \quad (2.2)$$

and V' is called a right \mathcal{A} -module; correspondingly one defines right \mathcal{A} -module algebras.

Just like the comultiplication is the dual operation to multiplication, *right* or *left coactions* are dual to left or right actions, respectively. A left coaction of a coalgebra \mathcal{C} (i.e., \mathcal{C} is equipped with a coproduct) on a vector space V is defined as a linear map

$$\Delta_{\mathcal{C}} : V \rightarrow \mathcal{C} \otimes V : \quad v \mapsto \Delta_{\mathcal{C}}(v) \equiv v^{(1)'} \otimes v^{(2)}, \quad (2.3)$$

such that

$$(\text{id} \otimes \Delta_{\mathcal{C}})\Delta_{\mathcal{C}} = (\Delta \otimes \text{id})\Delta_{\mathcal{C}}, \quad (\epsilon \otimes \text{id})\Delta_{\mathcal{C}} = \text{id}. \quad (2.4)$$

The prime on the first factor marks a left coaction. If \mathcal{C} is a Hopf algebra coacting on an algebra \mathcal{F} , we say that \mathcal{F} is a *right \mathcal{C} -comodule algebra* if $\Delta_{\mathcal{C}}(a \cdot b) = \Delta_{\mathcal{C}}(a) \cdot \Delta_{\mathcal{C}}(b)$ and $\Delta_{\mathcal{C}}(1) = 1 \otimes 1$, for all $a, b \in \mathcal{F}$. Similarly one defines right comodule algebras.

Now if the coalgebra \mathcal{C} is dual to an algebra \mathcal{A} in the sense of (1.32), then a left coaction of \mathcal{C} on V induces a right action of \mathcal{A} on V and vice versa, via

$$v \triangleleft x \equiv \langle x, v^{(1)'} \rangle v^{(2)}, \quad (2.5)$$

and right coactions induce left actions. More on these structures can be found in [41, 51].

For our purpose, we will consider left coactions of $\text{Fun}(G_q)$ on left comodule algebras $\text{Fun}(\mathcal{M}_q)$, which according to the above corresponds to right actions of $\mathcal{U} = U_q(g)$ on $\text{Fun}(\mathcal{M}_q)$. Notice that for quantum groups, a left \mathcal{U} -module algebra \mathcal{F} can always be transformed into a right \mathcal{U} -module algebra and vice versa using the (linear!) Cartan–Weyl involution: $a \triangleleft x \equiv \theta(x) \triangleright a$ for $a \in \mathcal{F}$ and $x \in \mathcal{U}$. Alternatively, one could use the antipode instead of θ , but this is a priori not compatible with the algebra structure of \mathcal{F} .

2.1.2 Quantum Spaces and Calculus as Comodule Algebras

First a word on the conventions. We have seen in section 1.1.4 that $\text{Fun}(G_{q^{-1}})$ is dual to $U_q(g)$. However most of the literature on quantum spaces uses $\text{Fun}(G_q)$, and

therefore we will do the same in this section. We may later have to replace q by q^{-1} when we make contact with $U_q(g)$.

Recall that $Fun(G_q)$ is the algebra generated by matrix elements A_j^i with relations (1.38)

$$\hat{R}_{mn}^{ik} A_j^m A_l^n = A_n^i A_m^k \hat{R}_{jl}^{nm}, \quad (2.6)$$

where $\hat{R}_{mn}^{ik} = R_{mn}^{ki}$ and R_{mn}^{ik} is the N -dimensional vector representation of \mathcal{R} . This is nothing but the statement that the \hat{R} -matrix commutes with the action of \mathcal{U} , in the dual picture. The explicit form of \hat{R} depends on the group and is given e.g. in [18]. Unless we are dealing with $Fun(GL_q(N))$, this has to be supplemented by additional relations corresponding to invariant bilinear forms or determinants (otherwise \mathcal{U} and $Fun(G_q)$ are not dual, cp. [41]).

Quantum Euclidean group and space We only consider the case of $Fun(SO_q(N))$ and its real forms in detail¹. In that case, the tensor product of 2 vector representations contains a trivial representation corresponding to the invariant bilinear form. This can be seen from the \hat{R} -matrix, which by virtue of section 1.2.2 decomposes into 3 projectors [18] $\hat{R}_{kl}^{ij} = (qP^+ - q^{-1}P^- + q^{1-N}P^0)_{kl}^{ij}$. The metric g_{ij} is then determined by $(P^0)_{kl}^{ij} = \frac{q^2-1}{(q^N-1)(q^{2-N}+1)} g^{ij} g_{kl}$, where $g_{ik} g^{kj} = \delta_i^j$. Explicitly,

$$g_{ij} = \delta_{i,j'} q^{-\rho_i}, \quad (2.7)$$

where $j' = N + 1 - i$ and ρ_i are the values of the Weyl vector $\tilde{\rho}$ in the vector representation. For $SO_q(N)$ with N odd, $\rho_i = (N/2-1, N/2-2, \dots, 1/2, 0, -1/2, \dots, 1-N/2)$. Furthermore, $D_j^i \equiv \delta_{i,j} q^{-2\rho_i} = g^{ik} g_{jk}$ generates the square of the antipode (see section 1.1.4; notice the replacement $q \rightarrow q^{-1}$ pointed out in 1.1.3). The last equality follows from Proposition 4.3.2.

In the language of coactions, invariance of g_{ij} becomes

$$g_{ij} A_k^i A_l^j = g_{kl}, \quad (2.8)$$

which must be imposed on $Fun(SO_q(N))$. In section 4.3, we will find a very interesting interpretation of g_{ij} , which will show various consistency conditions between g_{ij} and the \hat{R} -matrix. They can be used to show that (2.7) is consistent, in particular that the lhs is central in $Fun(SO_q(N))$. We refer to [18] or [44] for more details.

¹Here, the series B_n and D_n can be treated simultaneously.

Similarly, the tensor product of N vector representations contains a trivial representation corresponding to the totally antisymmetric tensor,

$$A_{j_1}^{i_1} \dots A_{j_N}^{i_N} \epsilon_q^{j_1 \dots j_N} = \epsilon_q^{i_1 \dots i_N}, \quad (2.9)$$

and ϵ_q also satisfies certain consistency conditions. Both g_{ij} and $\epsilon_q^{i_1 \dots i_N}$ depend analytically on q and reduce to the classical expressions as $q \rightarrow 1$.

Now (the algebra of functions on) *quantum Euclidean space* $Fun(E_q^N)$ [18] is generated by x^i with commutation relations

$$(P^-)_{kl}^{ij} x^k x^l = 0. \quad (2.10)$$

The center is generated by 1 and $r^2 = g_{ij} x^i x^j$. One can go further and define algebras of differential forms, derivatives, and so on, see [58, 44, 64]. The algebra of differential forms is defined by $(P^+)_{kl}^{ij} dx^k dx^l = 0$ and $g_{ij} dx^i dx^j = 0$, i.e.

$$dx^i dx^j = -q \hat{R}_{kl}^{ij} dx^k dx^l. \quad (2.11)$$

The epsilon-tensor is then determined by the unique top - (N-) form

$$dx^{i_1} \dots dx^{i_N} = \epsilon_q^{i_1 \dots i_N} dx^1 \dots dx^N \equiv \epsilon_q^{i_1 \dots i_N} d^N x. \quad (2.12)$$

One can introduce derivatives which satisfy

$$(P^-)_{kl}^{ij} \partial^k \partial^l = 0, \quad (2.13)$$

$$\partial^i x^j = g^{ij} + q(\hat{R}^{-1})_{kl}^{ij} x^k \partial^l, \quad (2.14)$$

and

$$\partial^i dx^j = q^{-1} \hat{R}_{kl}^{ij} dx^k \partial^l, \quad x^i dx^j = q \hat{R}_{kl}^{ij} dx^k x^l. \quad (2.15)$$

All this is consistent, and represents one possible choice. For more details, see e.g. [44].

It can be checked that all the above relations are preserved under the coaction of $Fun(SO_q(N))$

$$\begin{aligned} \Delta(x^i) &= A_j^i \otimes x^j \equiv x_{(1)}^i \otimes x_{(2)}^i, \\ \Delta(dx^i) &= A_j^i \otimes dx^j \end{aligned} \quad (2.16)$$

etc., in Sweedler - notation.

Finally, the *quantum sphere* S_q^{N-1} is generated by $t^i = x^i/r$ where r is central, so $g_{ij}t^it^j = 1$.

So far, we have not specified any reality structure, i.e. all the above spaces are complex. To define real quantum spaces, we have to impose a star-structure on $Fun(SO_q(N))$ and $Fun(E_q^N)$, i.e. an antilinear involution on these algebras. Again, one has to distinguish the cases of $q \in \mathbb{R}$ and $|q| = 1$. In this chapter, we will consider the Euclidean case, which corresponds to $q \in \mathbb{R}$. Later we will consider the Anti-de Sitter case, for $|q| = 1$.

So from now on $q \in \mathbb{R}$. Then there is a star-structure

$$\overline{A_j^i} = g^{jm} A_m^l g_{li} \quad (2.17)$$

extended as antilinear involution, which corresponds to $Fun(SO_q(N, \mathbb{R}))$ or $Fun(SO_q(N, \mathbb{R}))^2$. The antipode can then be written as

$$S(A_j^i) = \overline{A_i^j}. \quad (2.18)$$

On quantum Euclidean space, the corresponding involution is $\overline{x^i} = x^j g_{ji}$ [18], which is compatible with the left coaction of $Fun((S)O_q(N))$, i.e. $\overline{\Delta(x^i)} = \Delta(\overline{x^i})$. Even though the metric (2.7) looks unusual because we are working in a weight basis, this is indeed a Euclidean space. The extension of this involution to the differentials and derivatives is quite complicated [44], but this will not be necessary for our purpose.

Since $\overline{r^2} = r^2$, this also induces an involution on the quantum sphere S_q^{N-1} , which becomes the *Euclidean quantum sphere*³.

In this chapter, we will often write $SO_q(N) \equiv Fun(SO_q(N, \mathbb{R}))$ for this real ("compact") version of $Fun(SO_q(N))$, abusing an earlier convention in the dual picture. Similarly, we will write $O_q(N)$ if the determinant condition (2.9) is not imposed for the sake of generality.

2.2 Integration on Quantum Euclidean Space and Sphere

²These are C^* algebras [48].

³another C^* algebra.

2.2.1 Introduction

As a first application of this formalism, we will define invariant integrals of functions or forms over q - deformed Euclidean space and spheres in N dimensions.

In the simplest case of the quantum plane, such an integral was first introduced by Wess and Zumino [58]; see also [7]. In the case of quantum Euclidean space, the Gaussian integration method was proposed by a number of authors [19, 31]. However, it is tedious to calculate except in the simplest cases and its properties have never been investigated thoroughly; in particular, we point out that determining the class of integrable functions is a rather subtle issue.

In this chapter, we will give a different definition based on spherical integration in N dimensions and investigate its properties in detail [55]. Although this idea has already appeared in the literature [24], it has not been developed very far. We first show that there is a unique invariant integral over the quantum Euclidean sphere, and prove that it is positive definite and satisfies a cyclic property involving the D -matrix of $SO_q(N)$. The integral over quantum Euclidean space is then defined by radial integration, both for functions and N forms. One naturally obtains a large class of integrable functions. It turns out not to be determined uniquely by rotation and translation invariance (=Stokes theorem) alone; it is unique after e.g. imposing a general scaling law. It is positive definite as well and thus allows to define a Hilbertspace of square - integrable functions, and satisfies the same cyclic property. The cyclic property also holds for the integral of N and $N - 1$ -forms over spheres, which leads to a simple, truly noncommutative proof of Stokes theorem on Euclidean space with and without spherical boundary terms, as well as on the sphere. These proofs only work for $q \neq 1$, nevertheless they reduce to the classical Stokes theorem for $q \rightarrow 1$. This shows the power of noncommutative geometry.

Although only the case of quantum Euclidean space is considered here, the general approach is clearly applicable to other reality structures as well. In particular, we will later consider the case of quantum

Anti-de Sitter space, which is nothing but the quantum sphere S_q^4 with a suitable reality structure. As expected, an integral can be obtained from the Euclidean case by analytic continuation. We hope that this will eventually find applications e.g. to define actions for field theories on such noncommutative spaces.

The conventions are as in the previous section with $q \in \mathbb{R}$ except in some proofs.

2.2.2 Integral on the Quantum Sphere S_q^{N-1}

We first define a (complex - valued) integral $\langle f(t) \rangle_t$ of a function $f(t)$ over S_q^{N-1} . It should certainly be invariant under $O_q(N)$, which means

$$A_{j_1}^{i_1} \dots A_{j_n}^{i_n} \langle t^{j_1} \dots t^{j_n} \rangle_t = \langle t^{i_1} \dots t^{i_n} \rangle_t. \quad (2.19)$$

Of course, it has to satisfy

$$g_{i_l i_{l+1}} \langle t^{i_1} \dots t^{i_n} \rangle_t = \langle t^{i_1} \dots t^{i_{l-1}} t^{i_{l+2}} \dots t^{i_n} \rangle_t \quad \text{and} \quad (P^-)_{j_l j_{l+1}}^{i_l i_{l+1}} \langle t^{j_1} \dots t^{j_n} \rangle_t = 0 \quad (2.20)$$

We require one more property, namely that $I^{i_1 \dots i_n} \equiv \langle t^{i_1} \dots t^{i_n} \rangle_t$ is analytic in $(q-1)$, i.e. its Laurent series in $(q-1)$ has no negative terms (we can then assume that the classical limit $q=1$ is nonzero). These properties in fact determine the spherical integral uniquely: for n odd, one should define $\langle t^{i_1} \dots t^{i_n} \rangle_t = 0$, and

Proposition 2.2.1 *For even n , there exists (up to normalization) one and only one tensor $I^{i_1 \dots i_n} = I^{i_1 \dots i_n}(q)$ analytic in $(q-1)$ which is invariant under $O_q(N)$*

$$A_{j_1}^{i_1} \dots A_{j_n}^{i_n} I^{j_1 \dots j_n} = I^{i_1 \dots i_n} \quad (2.21)$$

and symmetric,

$$(P^-)_{j_l j_{l+1}}^{i_l i_{l+1}} I^{j_1 \dots j_n} = 0 \quad (2.22)$$

for any l . It can be normalized such that

$$g_{i_l i_{l+1}} I^{i_1 \dots i_n} = I^{i_1 \dots i_{l-1} i_{l+2} \dots i_n} \quad (2.23)$$

for any l . $I^{ij} \propto g^{ij}$.

An explicit form is e.g. $I^{i_1 \dots i_n} = \lambda_n(\Delta^{n/2} x^{i_1} \dots x^{i_n})$, where $\Delta = g_{ij} \partial^i \partial^j$ is the Laplacian (in either of the 2 possible calculi), and λ_n is analytic in $(q-1)$. For $q=1$, they reduce to the classical symmetric invariant tensors.

Proof The proof is by induction on n . For $n=2$, g^{ij} is in fact the only invariant symmetric (and analytic) such tensor.

Assume the statement is true for n , and suppose I_{n+2} and I'_{n+2} satisfy the above conditions. Using the uniqueness of I_n , we have (in shorthand - notation)

$$g_{12} I_{n+2} = f(q-1) I_n \quad (2.24)$$

$$g_{12} I'_{n+2} = f'(q-1) I_n \quad (2.25)$$

where the $f(q-1)$ are analytic, because the left - hand sides are invariant, symmetric and analytic. Then $J_{n+2} = f'I_{n+2} - fI'_{n+2}$ is symmetric, analytic, and satisfies $g_{12}J_{n+2} = 0$. It remains to show that $J = 0$.

Since J is analytic, we can write

$$J^{i_1 \dots i_n} = \sum_{k=n_0}^{\infty} (q-1)^k J_{(k)}^{i_1 \dots i_n}. \quad (2.26)$$

$(q-1)^{-n_0} J^{i_1 \dots i_n}$ has all the properties of J and has a well-defined, nonzero limit as $q \rightarrow 1$; so we may assume $J_{(0)} \neq 0$.

Now consider invariance,

$$J^{i_1 \dots i_n} = A_{j_1}^{i_1} \dots A_{j_n}^{i_n} J^{j_1 \dots j_n}. \quad (2.27)$$

This equation is valid for all q , and we can take the limit $q \rightarrow 1$. Then A_j^i generate the commutative algebra of functions on the classical Lie group $O(N)$, and J becomes $J_{(0)}$, which is just a classical tensor. Now $(P^-)_{j_l j_{l+1}}^{i_l i_{l+1}} J^{j_1 \dots j_n} = 0$ implies that $J_{(0)}$ is symmetric for neighboring indices, and therefore it is completely symmetric. With $g_{12}J = 0$, this implies that $J_{(0)}$ is totally traceless (classically!). But there exists no totally symmetric traceless invariant tensor for $O(N)$. This proves uniqueness. In particular, $I^{i_1 \dots i_n} = \lambda_n (\Delta^{n/2} x^{i_1} \dots x^{i_n})$ obviously satisfies the assumptions of the proposition; it is analytic, because in evaluating the Laplacians, only the metric and the \hat{R} - matrix are involved, which are both analytic. Statement (2.23) now follows because x^2 is central. \square

Such invariant tensors have also been considered in [19] (where they are called S), as well as the explicit form involving the Laplacian. Our contribution is a self-contained proof of their uniqueness. So $\langle t^{i_1} \dots t^{i_n} \rangle_t \equiv I^{i_1 \dots i_n}$ for even n (and 0 for odd n) defines the unique invariant integral over S_q^{N-1} , which thus coincides with the definition given in [24].

From now on we only consider $N \geq 3$ since for $N = 1, 2$, Euclidean space is undeformed. The following lemma is the origin of the cyclic properties of invariant tensors. For quantum groups, the square of the antipode is usually not 1. For $(S)O_q(N)$, it is generated by the D - matrix: $S^2 A_j^i = D_l^i A_k^l (D^{-1})_j^k$ where $D_l^i = g^{ik} g_{lk}$ (note that D also defines the quantum trace). Then

Lemma 2.2.2 *For any invariant tensor $J^{i_1 \dots i_n} = A_{j_1}^{i_1} \dots A_{j_n}^{i_n} J^{j_1 \dots j_n}$, $D_{l_1}^{i_1} J^{i_2 \dots l_1}$ is invariant too:*

$$A_{j_1}^{i_1} \dots A_{j_n}^{i_n} D_{l_1}^{j_1} J^{j_2 \dots l_1} = D_{l_1}^{i_1} J^{i_2 \dots l_1} \quad (2.28)$$

Proof From the above, (2.28) amounts to

$$(S^{-2}A_{j_1}^{i_1})A_{j_2}^{i_2}...A_{j_n}^{i_n}J^{j_2...j_nj_1} = J^{i_2...i_ni_1}. \quad (2.29)$$

Multiplying with $S^{-1}A_{i_1}^{i_0}$ from the left and using $S^{-1}(ab) = (S^{-1}b)(S^{-1}a)$ and $(S^{-1}A_{j_1}^{i_1})A_{i_1}^{i_0} = \delta_{j_1}^{i_0}$, this becomes

$$A_{j_2}^{i_2}...A_{j_n}^{i_n}J^{j_2...j_ni_0} = S^{-1}A_{i_1}^{i_0}J^{i_2...i_ni_1}. \quad (2.30)$$

Now multiplying with $A_{i_0}^{l_0}$ from the right, we get

$$A_{j_2}^{i_2}...A_{j_n}^{i_n}A_{i_0}^{l_0}J^{j_2...j_ni_0} = \delta_{i_1}^{l_0}J^{i_2...i_ni_1}. \quad (2.31)$$

But the (lhs) is just $J^{i_2...i_nl_0}$ by invariance and thus equal to the (rhs). \square

We can now show a number of properties of the integral over the sphere:

Theorem 2.2.3

$$\overline{\langle f(t) \rangle_t} = \langle \overline{f(t)} \rangle_t \quad (2.32)$$

$$\langle \overline{f(t)}f(t) \rangle_t \geq 0 \quad (2.33)$$

$$\langle f(t)g(t) \rangle_t = \langle g(t)f(Dt) \rangle_t \quad (2.34)$$

where $(Dt)^i = D_j^i t^j$. The last statement follows from

$$I^{i_1...i_n} = D_{j_1}^{i_1} I^{i_2...i_nj_1}. \quad (2.35)$$

Proof For (2.32), we have to show that $I^{j_n...j_1}g_{j_ni_n}...g_{j_1i_1} = I^{i_1...i_n}$. Using the uniqueness of I , it is enough to show that $I^{j_n...j_1}g_{j_ni_n}...g_{j_1i_1}$ is invariant, symmetric and normalized as I . So first,

$$\begin{aligned} A_{j_1}^{i_1}...A_{j_n}^{i_n} \left(I^{k_n...k_1} g_{k_nj_n}...g_{k_1j_1} \right) &= g_{l_1i_1}...g_{l_ni_n} \overline{A_{k_n}^{l_n}...A_{k_1}^{l_1}} I^{k_n...k_1} \\ &= \overline{A_{k_n}^{l_n}...A_{k_1}^{l_1}} I^{k_n...k_1} g_{l_1i_1}...g_{l_ni_n} \\ &= \left(I^{l_n...l_1} g_{l_ni_n}...g_{l_1i_1} \right). \end{aligned} \quad (2.36)$$

We have used that I is real (since g^{ij} and \hat{R} are real), and $A_{j_1}^{i_1}g_{k_1j_1} = g_{l_1i_1}\overline{A_{k_1}^{l_1}}$. The symmetry condition (2.22) follows from standard compatibility conditions between \hat{R} and g^{ij} , and the fact that \hat{R} is symmetric. The correct normalization can be seen easily using $g^{ij} = g_{ij}$ for q - Euclidean space.

To show positive definiteness (2.33), we use the observation made by [18] that

$$t^i \rightarrow A_j^i u^j \quad (2.37)$$

with $u^j = u_1 \delta_1^j + u_N \delta_N^j$ is an embedding $S_q^{N-1} \rightarrow Fun(O_q(N))$ for $u_1 u_N = (q^{(N-2)/2} + q^{(2-N)/2})^{-1}$, since $(P^-)^{ij}_{kl} u^k u^l = 0$ and $g_{ij} u^i u^j = 1$. In fact, this embedding also respects the star - structure if one chooses $u_N = u_1 q^{1-N/2}$ and real. Now one can write the integral over S_q^{N-1} in terms of the Haar - measure on the compact quantum group $O_q(N, \mathbb{R})$ [61, 48]. Namely,

$$\langle t^{i_1} \dots t^{i_n} \rangle_t = \langle A_{j_1}^{i_1} \dots A_{j_n}^{i_n} \rangle_A u^{j_1} \dots u^{j_n} \equiv \langle A_{\underline{j}}^{\underline{i}} \rangle_A u^{\underline{j}}, \quad (2.38)$$

(in short notation) since the Haar - measure $\langle \rangle_A$ is left (and right) - invariant $\langle A_{\underline{j}}^{\underline{i}} \rangle_A = A_{\underline{k}}^{\underline{i}} \langle A_{\underline{j}}^{\underline{k}} \rangle_A = \langle A_{\underline{k}}^{\underline{i}} \rangle_A A_{\underline{j}}^{\underline{k}}$ and analytic, and the normalization condition is satisfied as well. Then $\langle \overline{t^i} t^j \rangle_t = \langle \overline{A_{\underline{k}}^i} A_{\underline{r}}^j \rangle_A u^{\underline{k}} u^{\underline{r}}$ and for $f(t) = \sum f_{\underline{i}} t^{\underline{i}}$ etc.,

$$\begin{aligned} \langle \overline{f(t)} g(t) \rangle_t &= \overline{f_{\underline{i}} g_{\underline{j}}} \langle \overline{A_{\underline{k}}^i} A_{\underline{r}}^j \rangle_A u^{\underline{k}} u^{\underline{r}} = \langle \overline{f_{\underline{i}} A_{\underline{k}}^i} u^{\underline{k}} \rangle (g_{\underline{j}} A_{\underline{r}}^j u^{\underline{r}}) \rangle_A \\ &= \langle \overline{f(Au)} g(Au) \rangle_A. \end{aligned} \quad (2.39)$$

This shows that the integral over S_q^{N-1} is positive definite, because the Haar - measure over compact quantum groups is positive definite [61], cp. [11].

Finally we show the cyclic property (2.35). (2.34) then follows immediately. For $n = 2$, the statement is obvious: $g^{ij} = D_k^i g^{jk}$.

Again using a shorthand - notation, define

$$J^{12\dots n} = D_1 I^{23\dots n1}. \quad (2.40)$$

Using the previous proposition, we only have to show that J is symmetric, invariant, analytic and properly normalized. Analyticity is obvious. The normalization follows immediately by induction, using the property shown in proposition (2.2.1). Invariance of J follows from the above lemma. It remains to show that J is symmetric, and the only nontrivial part of that is $(P^-)_{12} J^{12\dots n} = 0$. Define

$$\tilde{J}^{12\dots n} = (P^-)_{12} J^{12\dots n}, \quad (2.41)$$

so \tilde{J} is invariant, antisymmetric and traceless in the first two indices (12), symmetric in the remaining indices (we will say that such a tensor has the ISAT property), and analytic. It is shown below that there is no such \tilde{J} for $q = 1$ (and $N \geq 3$). Then as

in proposition (2.2.1), the leading term of the expansion of \tilde{J} in $(q - 1)$ is classical and therefore vanishes, which proves that $\tilde{J} = 0$ for any q .

So from now on $q = 1$. We show by induction that $\tilde{J} = 0$. This is true for $n = 2$: there is no invariant antisymmetric traceless tensor with 2 indices (for $N \geq 3$). Now assume the statement is true for n even, and that $\tilde{J}^{12\dots(n+2)}$ has the ISAT property. Define

$$K^{12\dots n} = g_{(n+1),(n+2)} \tilde{J}^{12\dots(n+2)}. \quad (2.42)$$

K has the ISAT property, so by the induction assumption

$$K = 0. \quad (2.43)$$

Define

$$M^{145\dots(n+2)} = g_{23} \tilde{J}^{12\dots(n+2)} = \mathcal{S}_{14} M^{145\dots(n+2)} + \mathcal{A}_{14} M^{145\dots(n+2)} \quad (2.44)$$

where \mathcal{S} and \mathcal{A} are the classical symmetrizer and antisymmetrizer. Again by the induction assumption, $\mathcal{A}_{14} M^{145\dots(n+2)} = 0$ (it satisfies the ISAT property). This shows that M is symmetric in the first two indices $(1, 4)$. Together with the definition of M , this implies that M is totally symmetric. Further, $g_{14} M^{145\dots(n+2)} = g_{14} g_{23} \tilde{J}^{12\dots(n+2)} = 0$ because \tilde{J} is antisymmetric in $(1, 2)$. But then M is totally traceless, and as in proposition (2.2.1) this implies $M = 0$. Together with (2.43) and the ISAT property of \tilde{J} , it follows that \tilde{J} is totally traceless. So \tilde{J} corresponds to a certain Young tableaux, describing a larger - than - one dimensional irreducible representation of $O(N)$. However, \tilde{J} being invariant means that it is a trivial one - dimensional representation. This is a contradiction and proves $\tilde{J} = 0$.

□

Property (2.33) (which is also implied by results in [19], once the uniqueness of the invariant tensors is established) in particular means that one can now define the Hilbertspace of square - integrable functions on S_q^{N-1} . The same will be true for the integral on the entire Quantum Euclidean space.

The cyclic property (2.34) is a strong constraint on $I^{i_1\dots i_n}$ and could actually be used to calculate it recursively, besides its obvious interest in its own. An immediate consequence of (2.34) is $\langle f(Dt) \rangle_t = \langle f(t) \rangle_t$, which also follows from rotation invariance of the integral, because D is essentially the representation of the (exponential of the) Weyl vector of $\mathcal{U}_q(SO(N))$.

Notice that although it may not look like, (2.34) is consistent with conjugation: even though the D - matrix is real, we have

$$\overline{\bar{f}(Dt)} = f(D^{-1}t). \quad (2.45)$$

To see this, take $f(t) = t^i$; then the (lhs) becomes

$$\overline{D(\bar{t}^i)} = \overline{D(t^j g_{ji})} = \overline{D_k^j t^k g_{ji}} = \quad (2.46)$$

$$= D_k^j t^l g_{lk} g_{ji} = t^l g_{jl} g_{ji} = (D^{-1})_l^i t^l \quad (2.47)$$

using the cyclic property of g and $D_l^i = g_{ik} g_{lk}$, which is the (rhs) of the above.

2.2.3 Integral over Quantum Euclidean Space

It is now easy to define an integral over quantum Euclidean space. Since the invariant length $r^2 = g_{ij} x^i x^j$ is central, we can use its square root r as well, and write any function on quantum Euclidean space in the form $f(x^i) = f(t^i, r)$. We then define its integral to be

$$\langle f(x) \rangle_x = \langle \langle f(t, r) \rangle_t (r) \cdot r^{N-1} \rangle_r, \quad (2.48)$$

where $\langle f(t, r) \rangle_t (r)$ is a classical, analytic function in r , and $\langle g(r) \rangle_r$ is some linear functional in r , to be determined by requiring Stokes theorem. It is essential that this radial integral $\langle g(r) \rangle_r$ is really a functional of the *analytic continuation* of $g(r)$ to a function on the (positive) real line. Only then one obtains a large class of integrable functions, and this concept of integration over the entire space agrees with the classical one.

It will turn out that Stokes theorem e.g. in the form $\langle \partial_i f(x) \rangle_x = 0$ holds if and only if the radial integral satisfies the scaling property

$$\langle g(qr) \rangle_r = q^{-1} \langle g(r) \rangle_r. \quad (2.49)$$

This can be shown directly; we will instead give a more elegant proof later. This scaling property is obviously satisfied by an arbitrary superposition of Jackson - sums,

$$\langle f(r) \rangle_r = \int_1^q dr_0 \mu(r_0) \sum_{n=-\infty}^{\infty} f(q^n r_0) q^n \quad (2.50)$$

with arbitrary (positive) "weight" function $\mu(r) > 0$. The normalization can be fixed such that e.g. $\langle e^{-r^2} \rangle_r$ gives the classical result. If $\mu(r)$ is a delta - function, this is simply a Jackson - sum; for $\mu(r) = 1$, one obtains the classical radial integration

$$\langle f(r) r^{N-1} \rangle_r = \int_1^q dr_0 \sum_{n=-\infty}^{\infty} q^n (q^n r_0)^{N-1} f(q^n r_0) = \int_0^\infty dr r^{N-1} f(r). \quad (2.51)$$

This is the unique choice of $\mu(r)$ for which the scaling property (2.49) holds for any positive real number, not just for powers of q . We define $f(x^i)$ to be integrable (with respect to $\mu(r)$) if the corresponding radial integral in (2.48) is finite. We therefore obtain generally inequivalent integrals for different choices of $\mu(r)$, all of which satisfy Stokes theorem.

Let us try to compare the above definitions with the Gaussian approach. In that case, one does not resort to a classical integral, and determining the class of integrable functions seems to be rather subtle. The Gaussian integration procedure is based on the observation that the integral of functions of the type (polynomial)·(Gaussian) is uniquely determined by Stokes theorem (and therefore agrees with our definition for any normalized $\mu(r)$); one would then like to extend it to more general functions by a limiting process. Lacking a natural topology on the space of functions (i.e. formal power – series), this limiting process is however quite problematic. One way to see this is because there are actually many different inequivalent integrals labeled by $\mu(r)$, such a limiting process can only be unique on the (presumably small) class of functions on which the integral is independent of $\mu(r)$. Furthermore even classically, although one can calculate e.g. $\int \frac{1}{r^2+1} e^{-r^2}$ by expanding it "properly" (i.e. using pointwise or L^2 convergence) in terms of Hermite functions, if one tries to expand it formally e.g. in terms of $\{r^n e^{-r^2}\}$, one obtains a divergent sum of integrals. Thus the result may depend on the choice of basis and limiting procedure. It is not clear to the author how to properly integrate functions other than (polynomial)·(Gaussian) in the Gaussian sense, which would be very desirable, because that approach may be applied to some quantum spaces which do not have a central length element [31].

The properties of the integral over S_q^{N-1} generalize immediately to the Euclidean case, for any positive $\mu(r)$:

Theorem 2.2.4

$$\overline{\langle f(x) \rangle_x} = \langle \overline{f(x)} \rangle_x \quad (2.52)$$

$$\langle \overline{f(x)} f(x) \rangle_x \geq 0 \quad (2.53)$$

$$\langle f(x) g(x) \rangle_x = \langle g(x) f(Dx) \rangle_x, \quad (2.54)$$

and

$$\langle f(qx) \rangle_x = q^{-N} \langle f(x) \rangle_x \quad (2.55)$$

if and only if (2.49) holds.

Proof Immediately from theorem (2.2.3), (2.49) and (2.48), using $Dr = r$ and $\mu(r_0) > 0$. \square

(2.52) and (2.55) were already known for the special case of the Gaussian integral [19]. It was pointed out to me by G. Fiore that in this case, positivity was also shown in [20].

2.2.4 Integration of Forms

It turns out to be very useful to consider not only integrals over functions, but also over forms, just like classically. As was mentioned before, there exists a unique N - form $dx^{i_1} \dots dx^{i_N} = \epsilon_q^{i_1 \dots i_N} d^N x$, and we define

$$\int_x d^N x f(x) = \langle f(x) \rangle_x, \quad (2.56)$$

i.e. we first commute $d^N x$ to the left, and then take the integral over the function on the right. Then the two statements of Stokes theorem $\langle \partial_i f(x) \rangle_x = 0$ and $\int_x d\omega_{N-1} = 0$ are equivalent.

The following observation by Bruno Zumino [65] will be very useful: there is a one - form

$$\omega = \frac{q^2}{(q+1)r^2} d(r^2) = q \frac{1}{r} dr = dr \frac{1}{r} \quad (2.57)$$

where $rdx^i = qdx^i r$, which generates the calculus on quantum Euclidean space by

$$[\omega, f]_{\pm} = (1 - q)df \quad (2.58)$$

for any form f with the appropriate grading. It satisfies

$$d\omega = \omega^2 = 0. \quad (2.59)$$

We define the integral of a N - form over the sphere $r \cdot S_q^{N-1}$ with radius r by

$$\int_{r \cdot S_q^{N-1}} d^N x f(x) = \omega r^N \langle f(x) \rangle_t = dr r^{N-1} \langle f(x) \rangle_t, \quad (2.60)$$

which is a one - form in r , as classically. It satisfies

$$\int_{r \cdot S_q^{N-1}} q^N d^N x f(qx) = \int_{qr \cdot S_q^{N-1}} d^N x f(x) \quad (2.61)$$

where $(drf(r))(qr) = qdrf(qr)$. Now defining $\int_r drg(r) = \langle g(r) \rangle_r$, (2.56) can be written as

$$\int_x d^N x f(x) = \int_r \left(\int_{r \cdot S_q^{N-1}} d^N x f(x) \right). \quad (2.62)$$

The scaling property (2.49), i.e. $\int_x d^N x f(qx) = q^{-N} \int_x d^N x f(x)$ holds if and only if the radial integrals satisfies

$$\int_r dr f(qr) = q^{-1} \int_r dr f(r). \quad (2.63)$$

We can also define the integral of a $(N-1)$ form $\alpha_{N-1}(x)$ over the sphere with radius r :

$$\int_{r \cdot S_q^{N-1}} \alpha_{N-1} = \omega^{-1} \left(\int_{r \cdot S_q^{N-1}} \omega \alpha_{N-1} \right). \quad (2.64)$$

The ω^{-1} simply cancels the explicit ω in (2.60), and it reduces to the correct classical limit for $q = 1$.

The epsilon - tensor satisfies the cyclic property:

Proposition 2.2.5

$$\epsilon_q^{i_1 \dots i_N} = (-1)^{N-1} D_{j_1}^{i_1} \epsilon_q^{i_2 \dots i_N j_1}. \quad (2.65)$$

Proof Define

$$\kappa^{12 \dots N} = (-1)^{N-1} D^1 \epsilon_q^{23 \dots N1} \quad (2.66)$$

in shorthand - notation again. Lemma (2.2.2) shows that κ is invariant. $\kappa^{12 \dots N}$ is traceless and $(q-)$ antisymmetric in $(23 \dots N)$. Now $g_{12} \kappa^{12 \dots N} = 0$ because there exists no invariant, totally antisymmetric traceless tensor with $(N-2)$ indices for $q = 1$, so by analyticity there is none for arbitrary q . Similarly from the theory of irreducible representations of $SO(N)$ [59], $P^+_{12} \kappa^{12 \dots N} = 0$ where P^+ is the q - symmetrizer, $1 = P^+ + P^- + P^0$. Therefore $\kappa^{12 \dots N}$ is totally antisymmetric and traceless (for neighboring indices), invariant and analytic. But there exists only one such tensor up to normalization (which can be proved similarly), so $\kappa^{12 \dots N} = f(q) \epsilon_q^{12 \dots N}$. It remains to show $f(q) = 1$. By repeating the above, one gets $\epsilon_q^{12 \dots N} = (f(q))^N (\det D) \epsilon_q^{12 \dots N}$ (here $12 \dots N$ stands for the *numbers* $1, 2, \dots, N$), and since $\det D = 1$, it follows $f(q) = 1$ (times a N -th root of unity, which is fixed by the classical limit). \square

Now consider a k - form $\alpha_k(x) = dx^{i_1} \dots dx^{i_k} f_{i_1 \dots i_k}(x)$ and a $(N-k)$ - form $\beta_{N-k}(x)$. Then the following cyclic property for the integral over forms holds:

Theorem 2.2.6

$$\int_{r \cdot S_q^{N-1}} \alpha_k(x) \beta_{N-k}(x) = (-1)^{k(N-k)} \int_{q^{-k} r \cdot S_q^{N-1}} \beta_{N-k}(x) \alpha_k(q^N Dx) \quad (2.67)$$

where $\alpha_k(q^N Dx) = (q^N Ddx)^{i_1} \dots (q^N Ddx)^{i_k} f_{i_1 \dots i_k}(q^N Dx)$.

In particular, when α_k and β_{N-k} are forms on S_q^{N-1} , i.e. they involve only $dx^i \frac{1}{r}$ and t^i , then

$$\int_{S_q^{N-1}} \alpha_k(t) \beta_{N-k}(t) = (-1)^{k(N-k)} \int_{S_q^{N-1}} \beta_{N-k}(t) \alpha_k(Dt). \quad (2.68)$$

On Euclidean space,

$$\int_x \alpha_k(x) \beta_{N-k}(x) = (-1)^{k(N-k)} \int_x \beta_{N-k}(x) \alpha_k(q^N Dx) \quad (2.69)$$

if and only if (2.63) holds.

Notice that on the sphere, $d^N x f(t) = f(t) d^N x$.

Proof We only have to show that

$$\int_{r \cdot S_q^{N-1}} f(x) d^N x g(x) = \int_{r \cdot S_q^{N-1}} d^N x g(x) f(q^N Dx) \quad (2.70)$$

and

$$\int_{r \cdot S_q^{N-1}} dx^i \beta_{N-1}(x) = (-1)^{N-1} \int_{q^{-1} r \cdot S_q^{N-1}} \beta_{N-1}(x) (q^N Ddx)^i. \quad (2.71)$$

(2.70) follows immediately from (2.34) and $x^i d^N x = d^N x q^N x^i$.

To see (2.71), we can assume that $\beta_{N-1}(x) = dx^{i_2} \dots dx^{i_N} f(x)$. The commutation relations $x^i dx^j = q \hat{R}_{kl}^{ij} dx^k x^l$ are equivalent to

$$\begin{aligned} f(q^{-1}x) dx^j &= \mathcal{R}((dx^j)_{(a)} \otimes f_{(1)})(dx^j)_{(b)} (f(x))_{(2)} \\ &= (dx^j \triangleleft \mathcal{R}^1)(f(x) \triangleleft \mathcal{R}^2) \end{aligned} \quad (2.72)$$

where $\mathcal{R} = \mathcal{R}^1 \otimes \mathcal{R}^2$ is the universal \mathcal{R} for $SO_q(N)$, using its quasitriangular property and $\mathcal{R}(A_k^j \otimes A_l^i) = \hat{R}_{kl}^{ij}$. $f \triangleleft Y = \langle Y, f_{(1)} \rangle f_{(2)}$ is the right action induced by the left coaction (2.16) of an element $Y \in \mathcal{U}_q(SO(N))$. Now invariance of the integral implies

$$(dx^j \triangleleft \mathcal{R}^1) \triangleleft f(x) \triangleleft \mathcal{R}^2 \triangleright_t = dx^j \triangleleft f(x) \triangleright_t, \quad (2.73)$$

because $\mathcal{R}^1 \otimes \epsilon(\mathcal{R}^2) = 1$. Using this, (2.72), (2.61) and (2.60), the (rhs) of (2.71) becomes

$$\begin{aligned}
(-1)^{N-1} \int_{q^{-1}r \cdot S_q^{N-1}} \beta_{N-1}(x) q^N D_j^i dx^j &= (-1)^{N-1} D_j^i \int_{r \cdot S_q^{N-1}} dx^{i_2} \dots dx^{i_N} f(q^{-1}x) dx^j \\
&= (-1)^{N-1} D_j^i \epsilon^{i_2 \dots i_N j} \omega r^N < f(x) >_t \\
&= \epsilon^{i_2 \dots i_N} \omega r^N < f(x) >_t \\
&= \int_{r \cdot S_q^{N-1}} dx^i \beta_{N-1}(x), \tag{2.74}
\end{aligned}$$

using (2.65). This shows (2.71), and (2.68) follows immediately. (2.69) then follows from (2.63).

□

Another way to show (2.71) following an idea of Branislav Jurco [28] is to use

$$\int_{r \cdot S_q^{N-1}} (\alpha_k \triangleleft SY) \beta_{N-k} = \int_{r \cdot S_q^{N-1}} \alpha_k (\beta_{N-k} \triangleleft Y) \tag{2.75}$$

to move the action of \mathcal{R}^2 in (2.72) to the left picking up $\mathcal{R}^1 S \mathcal{R}^2$, which generates the inverse square of the antipode and thus corresponds to the D^{-1} - matrix. This approach however cannot show (2.34) or (2.54), because the commutation relations of functions are more complicated.

(2.67) shows in particular that the definition (2.64) is natural, i.e. it essentially does not matter on which side one multiplies with ω .

Now we immediately obtain *Stokes theorem* for the integral over quantum Euclidean space, if and only if (2.63) holds. Noticing that $\omega(q^N D x) = \omega(x)$, (2.69) implies

$$\begin{aligned}
\int_x d\alpha_{N-1}(x) &= \frac{1}{1-q} \int_x [\omega, \alpha_{N-1}]_{\pm} \\
&\propto \int_x \omega \alpha_{N-1} - (-1)^{N-1} \alpha_{N-1} \omega \\
&= \int_x (-1)^{N-1} \alpha_{N-1} \omega - (-1)^{N-1} \alpha_{N-1} \omega = 0 \tag{2.76}
\end{aligned}$$

On the sphere, we get as easily

$$\begin{aligned}
\int_{S_q^{N-1}} d\alpha_{N-2}(t) &\propto \int_{S_q^{N-1}} [\omega, \alpha_{N-2}]_{\pm} \\
&= \omega^{-1} \int_{S_q^{N-1}} \omega (\omega \alpha_{N-2} - (-1)^{N-2} \alpha_{N-2} \omega) = 0 \tag{2.77}
\end{aligned}$$

using (2.68) and $\omega^2 = 0$.

It is remarkable that these simple proofs only work for $q \neq 1$, nevertheless the statements reduce to the classical Stokes theorem for $q \rightarrow 1$. This shows the power of the q - deformation technique.

One can actually obtain a version of Stokes theorem with spherical boundary terms. Define

$$\int_{q^k r_0}^{q^l r_0} \omega f(r) = \int_{q^k r_0}^{q^l r_0} dr \frac{1}{r} f(r) = (q-1) \sum_{n=k}^{l-1} f(r_0 q^n), \quad (2.78)$$

which reduces to the correct classical limit, because the (rhs) is a Riemann sum.

Define

$$\int_{q^k r_0 \cdot S_q^{N-1}}^{q^l r_0 \cdot S_q^{N-1}} \alpha_N(x) = \int_{q^k r_0}^{q^l r_0} \left(\int_{r \cdot S_q^{N-1}} \alpha_N(x) \right), \quad (2.79)$$

For $l \rightarrow \infty$ and $k \rightarrow -\infty$, this becomes an integral over Euclidean space as defined before. Then

$$\begin{aligned} \int_{q^k r_0 \cdot S_q^{N-1}}^{q^l r_0 \cdot S_q^{N-1}} d\alpha_{N-1} &= \frac{1}{1-q} \int_{q^k r_0}^{q^l r_0} \left(\int_{r \cdot S_q^{N-1}} \omega \alpha_{N-1} - (-1)^{N-1} \alpha_{N-1} \omega \right) \\ &= \frac{1}{1-q} \int_{q^k r_0}^{q^l r_0} \left(\int_{r \cdot S_q^{N-1}} \omega \alpha_{N-1} - \int_{qr \cdot S_q^{N-1}} \omega \alpha_{N-1} \right) \\ &= \int_{q^l r_0 \cdot S_q^{N-1}} \alpha_{N-1} - \int_{q^k r_0 \cdot S_q^{N-1}} \alpha_{N-1}. \end{aligned} \quad (2.80)$$

In the last line, (2.60), (2.64) and (2.78) was used.

2.3 Quantum Anti-de Sitter Space

Let us first review the classical Anti-de Sitter space (AdS space), which is a 4-dimensional manifold with constant curvature and signature $(+, -, -, -)$. It can be embedded as a hyperboloid into a 5-dimensional flat space with signature $(+, +, -, -, -)$, by

$$z_0^2 + z_4^2 - z_1^2 - z_2^2 - z_3^2 = R^2, \quad (2.81)$$

where R will be called the "radius" of the AdS space. Similarly we will consider the 2-dimensional version, defined by $z_0^2 + z_2^2 - z_1^2 = R^2$ (of course there is also a

3-dimensional case). The symmetry group (isometry group) of this space is $SO(2, 3)$ resp. $SO(2, 1)$, which plays the role of the Poincaré group. In fact, the Poincaré group can be obtained from $SO(2, 3)$ by a contraction, see e.g. [36].

This space has some rather peculiar features: First, its time-like geodesics are finite and closed. In particular, time "translations" are a $U(1)$ subgroup of $SO(2, 3)$. The space-like geodesics are unbounded. Furthermore the causal structure is somewhat complicated, but we will not worry about these issues here. With the goal in mind to eventually formulate a quantum field theory on a quantized version of "some" Minkowski-type spacetime, there are several reasons why we choose to work with this space and not e.g. with de Sitter space (corresponding to $SO(1, 4)$) or flat Minkowski space. First, $SO(2, 3)$ has unitary *positive-energy* representations corresponding to all elementary particles, as opposed to $SO(1, 4)$ [22], and it allows supersymmetric extensions [63]. Second, the seemingly simpler case of flat Minkowski space is actually mathematically more difficult, because the classical Poincaré group is not semi-simple, and the theory of quantum Poincaré groups is not as well developed as in the case of semi-simple quantum groups. But the main justification comes a posteriori, namely from the existence of *finite-dimensional* unitary representations of $SO_q(2, 3)$ for any spin at roots of unity, and some very encouraging results towards a formulation of gauge theories (=theories of massless particles, strictly speaking) in this framework, which will be presented below.

The heavy emphasis on group theory seems justified as a powerful guideline through the vast area of noncommutative geometry.

2.3.1 Definition and Basic Properties

Quantum Anti-de Sitter space (q-AdS space) will be defined as a real form of the complex quantum sphere S_q^4 defined above, with an (co)action of $SO_q(2, 3)$ which is a real form of $Fun(SO_q(5))$ resp. $U_q(so(5))$. Therefore the algebra of the coordinates $t^i \equiv x^i/r$ is

$$(P^-)^{ij}_{kl} t^k t^l = 0, \quad (2.82)$$

$$t \cdot t \equiv g_{kl} t^k t^l = 1. \quad (2.83)$$

For $|q| = 1$, consider the reality structure

$$\overline{t^i} = -(-1)^{E_i} t^j g_{ji} \quad (2.84)$$

extended as an antilinear algebra–automorphism. Here $E_i = (1, 0, 0, 0, -1)$ for $i = 1, 2, \dots, 5$ (resp. $E_i = (1, 0, -1)$ in the 2–dimensional case), which will turn out to be the eigenvalues of energy in the vector representation. It is easy to check that indeed $\overline{t \cdot t} = t \cdot t$. Correspondingly on $Fun(SO_q(2, 3))$, one can consider the reality structure

$$\overline{A_j^i} = (-1)^{E_i + E_j} g^{jm} A_m^l g_{li}, \quad (2.85)$$

extended as an antilinear algebra–automorphism. The fact that $\overline{(\cdot)}$ does not reverse the order is not a problem, since we will not consider the t^i as operators, only as ”coordinate functions” which will mainly be used in integrals, e.g. to write down Lagrangians. In fact, in quantum field theory the coordinates are not considered as operators on a Hilbert space. Thus this reality structure on q-AdS space has mainly illustrative character; some reality properties of the integral below however will be used to show hermiticity of interaction Lagrangians (if one would consider the t^i as operators on a space of functions on q-AdS space, the adjoint could be calculated from a positive–definite inner product, and would *not* be given by this reality structure). Observables like energy etc. do exist in our approach, in particular elements in the Cartan subalgebra of $U_q(SO(2, 3))$ which has a suitable reality structure. This is one of the reasons why we prefer to work with \mathcal{U} instead of $Fun(g)$.

To introduce proper units, define

$$y^i \equiv t^i R, \quad (2.86)$$

$$y \cdot y = y^i y^j g_{ij} = R^2 \quad (2.87)$$

for a constant⁴ $R \in \mathbb{R}_{>0}$.

So from now on $|q| = 1$. It is easy to see that (2.84) indeed corresponds to Anti–de Sitter space for $q = 1$: consider $R^2 = y \cdot y = y^i y^j g_{ij} = y^1 y^5 + y^2 y^4 + y^3 y^3 + y^4 y^2 + y^5 y^1$, and introduce real variables z^i by $y_1 = \frac{z^0 + iz^4}{\sqrt{2}}$, $y_5 = \frac{z^0 - iz^4}{\sqrt{2}}$, $y_2 = i \frac{z^1 + iz^3}{\sqrt{2}}$, $y_4 = i \frac{z^1 - iz^3}{\sqrt{2}}$, $y^3 = iz^2$. Plugging this into (2.87) gives the classical AdS space. This also shows that $E_i = (1, 0, 0, 0, -1)$ is indeed the energy (in suitable units), and similarly for the 2–dimensional version.

There are other possible reality structures which could define an AdS space for $|q| = 1$, such as $\overline{t^i}^b = -t^i$ and $\overline{A_j^i}^b = A_j^i$ extended as an antilinear involution. This is however not compatible with the identification of the energy in $U_q(so(2, 3))$ which is acting on it. It will nevertheless be useful in some calculations involving the integral.

⁴ R is different from r , which has nontrivial commutation relations with forms.

Integration. One can define an integral $\langle t^{i_1} \dots t^{i_n} \rangle_t$ on q-AdS space by analytic continuation in q from the integral over the Euclidean sphere S_q^4 (This clearly corresponds to the Wick rotation in QFT). It trivially satisfies the same algebraic properties as in the Euclidean case, and is compatible with both reality structures on AdS space:

Lemma 2.3.1 For $|q| = 1$,

$$\overline{\langle t^{i_1} \dots t^{i_n} \rangle_t} = \langle \overline{t^{i_1} \dots t^{i_n}} \rangle_t = \langle t^{i_n} \dots t^{i_1} \rangle_t \quad (2.88)$$

Proof Define $J^{i_n \dots i_1}(q) \equiv I^{i_1 \dots i_n}(q^{-1})$; then for $|q| = 1$, $J^{i_n \dots i_1} = (I^{i_1 \dots i_n})^*$, since q is the only complex quantity in the explicit formula in Proposition 2.2.1. Applying the above $\overline{(\quad)}^b$ to the statement of invariance (2.21), one gets $A_{j_n}^{i_n} \dots A_{j_1}^{i_1} J^{j_1 \dots j_n} = J^{i_1 \dots i_n}$. Now from a slightly generalized Proposition 2.2.1 where (2.21) and (2.22) are required only for $|q| = 1$, it follows that $J^{i_n \dots i_1}(q) = I^{i_1 \dots i_n}(q)$, since J and I are analytic in q . Alternatively, one can consider the anti-algebra automorphism $\rho(A_j^i) = A_j^i$, $\rho(q) = q^{-1}$, where q is treated as a formal variable.

Now (2.88) follows from (2.32). \square

At first sight, it may not look sensible to define an integral of polynomials on a noncompact space. However we are really interested in the case of roots of unity, where the analog of "square-integrable functions" are indeed obtained as (quotients of) polynomials, as explained in the following sections. The normalization has to be refined somewhat at roots of unity, and at this point, we make no statement on positivity.

2.3.2 Commutation Relations and Length Scale

Let us write down the algebra of coordinate functions on q-AdS space explicitly. This can be obtained from the Euclidean case [18]. In the 2-dimensional case one finds

$$qy^3y^1 - q^{-1}y^1y^3 = (q^{1/2} - q^{-1/2})R^2, \quad (2.89)$$

where y^2 is eliminated by the constraint $y \cdot y = R^2$. In 4 dimensions we find

$$\begin{aligned} y^i y^{i+k} &= qy^{i+k} y^i \quad \text{if } k > 0 \text{ and } 2i + k \neq 6, \\ qy^5 y^1 - q^{-1} y^1 y^5 &= \frac{q^{1/2} - q^{-1/2}}{q - 1 + q^{-1}} R^2 \end{aligned}$$

$$\begin{aligned}
qy^4y^2 - q^{-1}y^2y^4 &= (1 - q^2)y^5y^1 + q \frac{q^{1/2} - q^{-1/2}}{q - 1 + q^{-1}} R^2 \\
&= (q^{-2} - 1)y^1y^5 + q^{-1} \frac{q^{1/2} - q^{-1/2}}{q - 1 + q^{-1}} R^2
\end{aligned} \tag{2.90}$$

where y^3 is eliminated. The important point here is that these relations are inhomogeneous, and therefore contain an *intrinsic length scale*

$$L_0 \equiv \sqrt{|q^{1/2} - q^{-1/2}|} R; \tag{2.91}$$

notice that $(q - 1 + q^{-1}) \approx 1$, having in mind that q should be very close to 1. Since $|q| = 1$, $q = e^{2\pi i h}$ with h a small number (in fact $h = \frac{1}{m}$, as we will see below). Then $L_0 \approx \sqrt{2\pi h} R$. Also, notice that L_0 is much bigger than $|q - q^{-1}| R$ which one might have expected naively (and which will show up later). In flat Euclidean quantum space for example, the commutation relations are homogeneous, and no length scale appears.

To make these commutation relations more transparent, one can approximate them by $[y^5, y^1] = iL_0^2$ and $[y^4, y^2] = iL_0^2$. As in Quantum Mechanics, this means that the *geometry is classical for scales $\gg L_0$, and non-classical for scales $< L_0$* . Strictly speaking, this is only heuristic since the reality structure on the coordinates is not a standard star structure. However it is clear that there really *is* such a scale, and in the compact (Euclidean) version, the argument is indeed rigorous.

This is very satisfactory, and the way it should be if this is to find applications in high energy physics. Being extremely optimistic, one is tempted to identify L_0 with the Planck scale, where one expects the classical behaviour of space-time to break down. Of course, there is no justification for this so far. It means that q has to be *very* close to one. These considerations are continued in section 3.2.5.

Chapter 3

The Anti-de Sitter Group and its Unitary Representations

3.1 The Classical Case

3.1.1 $SO(2, 3)$ and $SO(2, 1)$

The classical AdS group is $SO(2, 3)$ resp. $U(so(2, 3))$, which is a real form of $U(so(5, \mathbb{C}))$ and plays the role of the Poincaré group.

The Cartan matrix for its rank 2 Lie algebra B_2 is

$$A_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad (3.1)$$

so $d_1 = 1, d_2 = 1/2$, to have the standard physics normalization. The weight diagrams of the vector representation V_5 and the spinor representation V_4 are given in figure 3.1 for illustration; the adjoint V_{10} is 10-dimensional. The Weyl vector is $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \frac{3}{2}\alpha_1 + 2\alpha_2$.

According to (1.22), we define $h_1 = H_1, h_2 = \frac{1}{2}H_2, e_{\pm 1} = X_1^{\pm}$, and $e_{\pm 2} = \sqrt{\frac{1}{2}}X_2^{\pm}$. Now one can obtain a Cartan–Weyl basis corresponding to all the roots, as explained in section 1.1.4. We choose a slightly different labeling here in order to have (essentially) the same conventions as in [36]. Using the longest element of the Weyl group $\omega = \tau_1\tau_2\tau_1\tau_2$, define $\beta_1 = \alpha_1, \beta_2 = \sigma_1\sigma_2\sigma_1\alpha_2 = \alpha_2, \beta_3 = \sigma_1\alpha_2 = \alpha_1 + \alpha_2, \beta_4 = \sigma_1\sigma_2\alpha_1 = \alpha_1 + 2\alpha_2$, see figure 3.1. The corresponding Cartan–Weyl basis of root vectors is $e_3 = [e_2, e_1], e_{-3} = [e_{-1}, e_{-2}], h_3 = h_1 + h_2$ and $e_4 = [e_2, e_3], e_{-4} = [e_{-3}, e_{-2}],$ and $h_4 = h_1 + 2h_2$. Alternatively one can use the

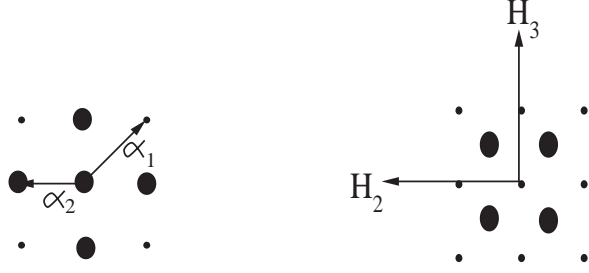


Figure 3.1: Vector and spinor representations of $SO(2, 3)$

braid-group action (1.46) which of course works for the classical case as well, i.e. $\{e_{\pm 1}, e_{\pm 3}, e_{\pm 4}, e_{\pm 2}\} = \{e_{\pm 1}, T_1 e_{\pm 2}, T_1 T_2 e_{\pm 1}, T_1 T_2 T_1 e_{\pm 2}\}$. Up to signs, this agrees with the basis used in [36].

The reality structure and the identification of the usual generators of the Poincare group in the limit $R \rightarrow \infty$ can be obtained by considering the algebra of generators leaving the metric invariant, see e.g. [22, 36]. It turns out that the following reality structure corresponds to $SO(2, 3)$:

$$\overline{H_i} = H_i, \quad \overline{X_1^+} = -X_1^-, \quad \overline{X_2^+} = X_2^-, \quad (3.2)$$

for $i = 1, 2$. Then

$$\overline{e_1} = -e_{-1}, \quad \overline{e_2} = e_{-2}, \quad \overline{e_3} = -e_{-3}, \quad \overline{e_4} = -e_{-4}. \quad (3.3)$$

We identify the weights of the vector representation $(y^1, y^2, y^3, y^4, y^5)$ to be $(\beta_3, \beta_2, 0, -\beta_2, -\beta_3)$, see figure 3.1. Then $\{e_{\pm 2}, h_2\}$ is a compact $SU(2)$ subalgebra which acts only on the spacial variables z^1, z^2, z^3 in AdS space (2.81). It corresponds to spatial rotations, and we will sometimes write

$$J_z \equiv h_2 \quad (3.4)$$

to indicate that it can be interpreted as a component of angular momentum. Furthermore $\{e_{\pm 3}, h_3\}$ is a noncompact $SO(2, 1)$ subalgebra acting on z^0, z^2 and z^4 . This is nothing but a 2 dimensional AdS group, and

$$E \equiv h_3 \quad (3.5)$$

is the energy since it generates rotations in the z^0, z^4 -plane. Then $E_i = (1, 0, 0, 0, -1)$ as in (2.85), and ρ_i is as given in section 2.1.2 for the Euclidean case. The reality structure (3.2) on $U(so(2, 3))$ can now be written as

$$\bar{x} \equiv (-1)^E \bar{x}^c (-1)^E, \quad (3.6)$$

where \bar{x}^c was defined in (1.73) for $x \in U(so(2, 3))$.

I want to give a brief explanation for the reality structure of $SO(2, 1)$. Let $X = \begin{pmatrix} a & b+c \\ b-c & -a \end{pmatrix}$, and $L \in SL(2, \mathbb{R})$. Then $\det(X) = -a^2 - b^2 + c^2$ is the quadratic form on 2-dimensional AdS space, which is invariant under $X \rightarrow L^{-1}XL$. Therefore $SL(2, \mathbb{R}) = SO(2, 1)$, at least locally. Now for $K_1 \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $K_2 \equiv \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $K_3 \equiv \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $K_{\pm} = K_1 \pm iK_2$, then $\{K_{\pm}, K_3\}$ is a $su(2)$ Lie algebra. Furthermore $K_a \equiv iK_1, K_b \equiv iK_2$ are purely imaginary, and therefore $L = \exp(i(\alpha_a K_a + \alpha_b K_b + \alpha_3 K_3)) \in SL(2, \mathbb{R})$ for real parameters $\alpha_{a,b,3}$. Now in a *unitary* representation of $SL(2, \mathbb{R})$ (which will be infinite-dimensional), $L^\dagger = L^{-1}$, so $K_{a,b,3}^\dagger = K_{a,b,3}$. But this means that $K_{\pm}^\dagger = -K_{\mp}$, and $K_3^\dagger = K_3$.

3.1.2 Unitary Representations, Massless Particles and BRST from a Group Theoretic Point of View

Let us briefly discuss the classical irreducible unitary representations of $SO(2, 3)$ corresponding to elementary particles. Of course, they are all infinite-dimensional.

The most important unitary positive-energy irreducible representations are lowest-weight representations $V_{(\lambda)}$ with lowest weight $\lambda = E_0\beta_3 - s\beta_2 \equiv (E_0, s)$ for any E_0 and s such that $E_0 \geq s + 1$, and both integer or both half integer (i.e. λ is integral, see section 1.2)) [22]. Unitarity will in fact follow from the quantum case. Then s is the spin of an elementary particle with rest energy E_0 . For example, a "scalar field" has $s = 0$ and $E_0 \geq 1$, see figure 3.2; it can be realized in the space of functions $f(y)$ on AdS space.

These representations have only discrete weights, nevertheless they become the usual irreps of the Poincare group in the limit $R \rightarrow \infty$, with appropriate rescaling.

There also exist remarkable unitary irreps with non-integral weights and all multiplicities equal to one, namely the so-called Dirac singletons "Di" for $\lambda = (1, 1/2)$ and "Rac" for $\lambda = (1/2, 0)$ [14]. While it is not clear if they could be of importance

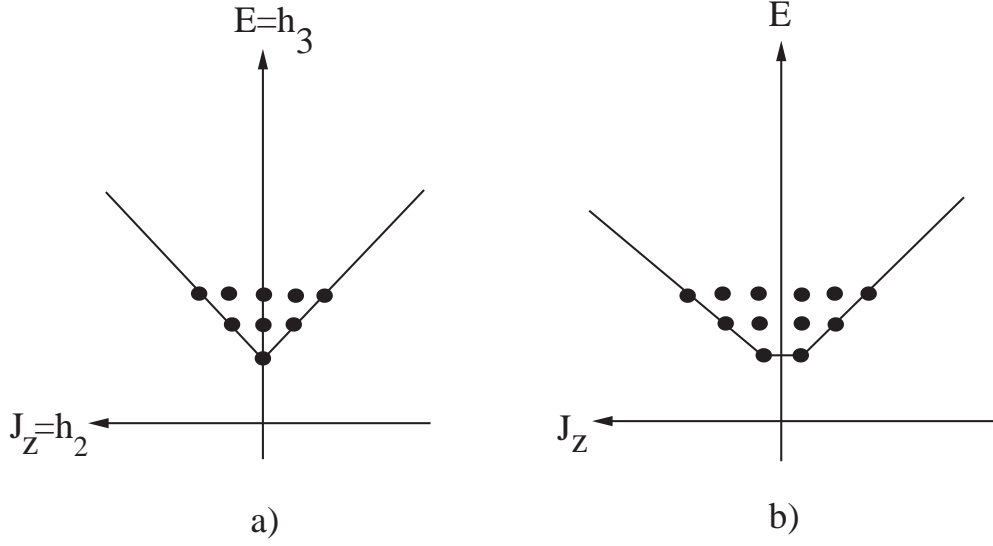


Figure 3.2: a) Scalar field and b) Spinor field. The vertical axis is energy, and the horizontal axis is a component of angular momentum.

in a theory of elementary particles, we will pursue them nevertheless. (We will also encounter some more representations with non-integral weights in the root of unity case. For a more general (classical) discussion, see [22].)

The massless case should be defined as $E_0 = s + 1$, cp. [22]. In this case, the rest energy E_0 is the smallest possible for a unitary representation of given spin. For $s \geq 1$, it is qualitatively different from the massive case with the same s : the lowest-weight representations, which are irreducible in the massive cases, develop an invariant subspace of "pure gauge" states with lowest weight $(E_0 + 1, s - 1)$. The representations however do not split into the direct sum of "pure gauges" plus the rest, i.e. they are not completely reducible. This means that there is no complete covariant gauge fixing, and to get rid of them and obtain a unitarizable, irreducible representation as required in a quantum theory, one *has* to factor them out. They are always null as we will see.

This corresponds precisely to the classical phenomenon in gauge theories, which ensures that the massless photon, graviton etc. have only their appropriate number of degrees of freedom. In general, the concept of mass in Anti-de Sitter space is not as clear as in flat space. Also notice that while "at rest" there are actually still $2s + 1$ states, the representation is nevertheless reduced by one irrep of spin $s - 1$.

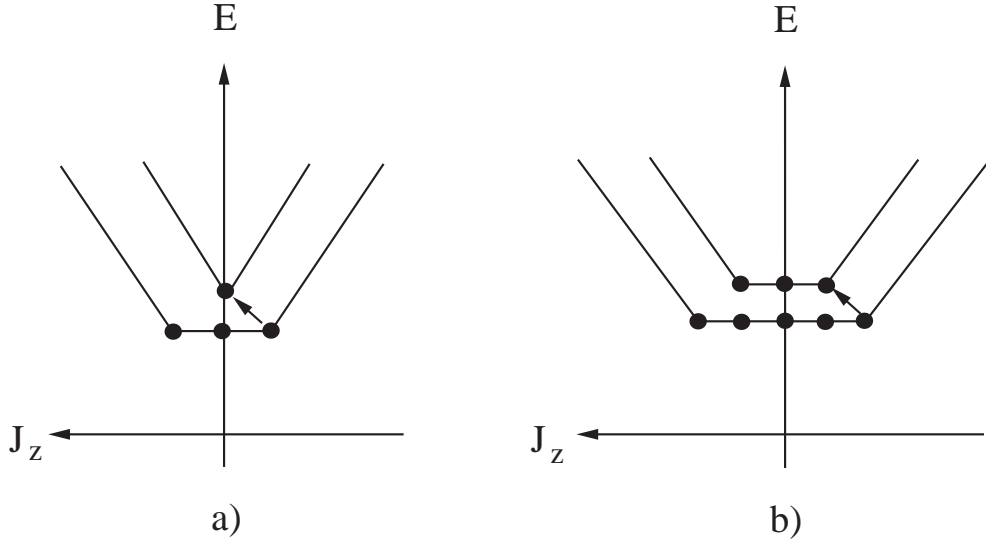


Figure 3.3: a) Photon and b) Graviton, with their "pure gauge" subspaces.

The massless representations for spin 1 ("vector field", "photon") and spin 2 ("graviton") are shown in figure 3.3, with their pure gauge subspaces. There are arrows (indicating the group action) into the subspace, but not out of it.

To understand the connection with the usual formalism, let us consider the spin 1 case in more detail. Spin one particles are usually described by one-forms, i.e. $A(y) = \sum A^i(y)dy^i$ in the natural embedding of AdS space, where "automatically" $\sum g_{ij}y^i dy^j = 0$. From a group-theoretic point of view, it would be more natural (and it is in fact unavoidable on q-AdS space) to consider unconstrained one-forms $A = \sum A^i(x)dx^i$, i.e. including the "radial" component, where x^i are the coordinates of the underlying 5-dimensional flat space. Such a general one-form is an element of $(\oplus_{E_0} V_{(E_0,0)}) \otimes V_5$ and vice versa, where $(\oplus_{E_0} V_{(E_0,0)})$ is a space of functions on AdS space spanned by the (unitary) scalar fields $V_{(E_0,0)}$, and V_5 is the 5-dimensional vector representation.

It is easy to see [22] that as representations,

$$V_{(E_0,0)} \otimes V_5 = V_{(E_0,1)} \oplus V_{(E_0+1,0)} \oplus V_{(E_0-1,0)}, \quad (3.7)$$

see figure 3.4 . $V_{(E_0,1)}$ is a vector field, $V_{(E_0+1,0)}$ is the space of "radial" one-forms $A^R(y)dR$ which is usually not considered in the flat case, and $V_{(E_0-1,0)}$ is what is usually called "longitudinal" modes, which can be killed by the constraint $d \star A(y) = 0$

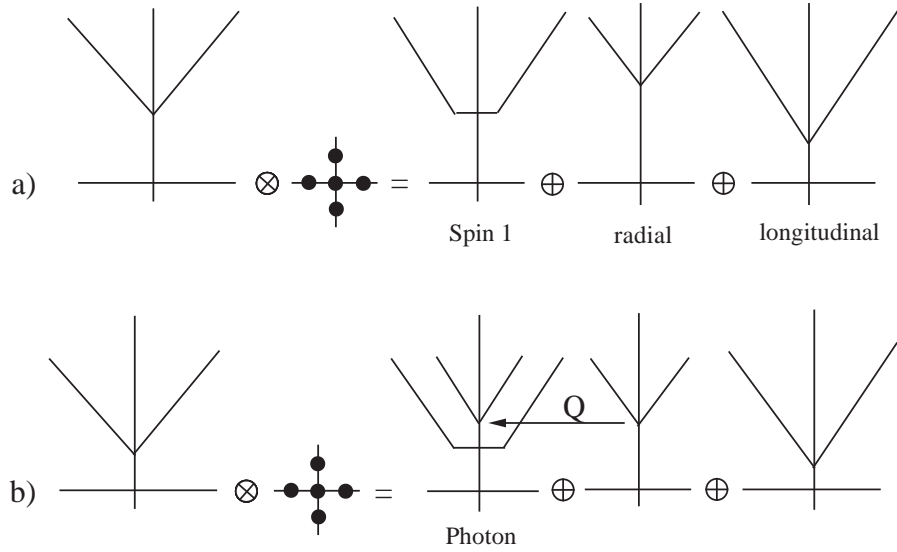


Figure 3.4: One-forms from a group theoretic point of view: a) massive and b) massless case, with BRST operator Q

("Lorentz gauge") where $*A(y)$ is the Hodge dual of $A(y)$ (in 4 dimensions; notice that $d * (V_{(E_0,1)}) \equiv 0$, since $d * A$ is a scalar). In fact, the $V_{(E_0-1,0)}$ part *has* to be discarded, since it would lead to negative norm states upon canonical quantization.

In the massless case $E_0 = 2$, $V_{(E_0,1)}$ has a null subspace of pure gauges (consisting of fields $A(y) = d\Lambda(y)$) which is isomorphic to $V_{(E_0+1,0)}$, and must be factored out. The essential and nontrivial point in a gauge theory is to show that the "pure gauge" subspaces do indeed decouple, so that they can *consistently* be factored out. Generally in QFT, this is best done using a BRST operator Q , which has the following characteristic properties:

- 1) The space of pure gauges is the image of Q (at ghost number 0)
- 2) Q commutes with the S - matrix, the action, etc.
- 3) $Q^2 = 0$

Then the physical Hilbert space can then be defined as the cohomology of Q at ghost number 0, i.e.

$$\mathcal{H}_{phys} = \{Q = 0\} / \text{Im}(Q)|_{gh \neq 0}, \quad (3.8)$$

where $Im(Q)$ is the image of Q , and 2) guarantees that this is consistent, i.e. $SQ(\dots) = Q(\dots)$. In the standard formulation for photons, $\{Q = 0\}$ also implies the constraint $d * A = 0$ (on the Hilbert space at ghost number 0), but this could as well be imposed by hand.

Now notice that in the AdS case (3.7), the radial components of a one-form and the subspace of pure gauges are isomorphic, and it is tempting to define an intertwiner Q from the former to the latter. On $V_{(E_0,1)}$ and $V_{(E_0-1,0)}$ in (3.7), define Q to be 0. Then Q acting on A indeed satisfies all the properties 1) to 3) of a BRST operator, and the radial component of A plays the role of a ghost; it is indeed a scalar, and anticommuting as a one – form.

Notice that we have only one ghost as opposed to 2 in the usual formulation, and accordingly $\{Q = 0\}$ does not constrain the longitudinal modes to vanish (this has to be imposed in addition). So this Q does not correspond precisely to the standard BRST operator in an abelian gauge theory¹. Nevertheless we will take the point of view that the above properties 1) to 3) are the characteristic ones, and call our Q a BRST operator as well. Actually, we will relax the requirement $Q^2 = 0$ in the most general setting (see theorem 4.1.2), but it will hold on the sectors of representations relevant to elementary particles.

Thus a BRST operator provides a way to define theories of massless elementary particles, i.e. massless unitary irreps. I consider this to be the essential feature of a (abelian) "gauge theory", and not some kind of "local gauge invariance" which is unphysical anyway.

Let us try to see if and how all this works in the q -deformed case.

3.2 The Quantum Anti-de Sitter Group at Roots of Unity

The quantum Anti-de Sitter group is simply $SO_q(2, 3) \equiv U_q(so(2, 3))$ as explained in section 1.1 for $|q| = 1$, with the same reality structure (3.2) as in the undeformed case, i.e.

$$\overline{x} = (-1)^{-E} \overline{x^c} (-1)^E. \quad (3.9)$$

This is consistent with all the properties (1.73) to (1.78), as explained in section 1.1.5. We do not consider $q \in \mathbb{R}$, because we will mainly be interested in the roots of unity

¹I wish to thank B. Morariu for discussions on this

case. The root vectors in the quantum case are defined by the braid group action as in section 3.1.1, now using the formulas (1.46). We obtain

$$\begin{aligned} e_3 &= q^{-1}e_2e_1 - e_1e_2, & e_{-3} &= qe_{-1}e_{-2} - e_{-2}e_{-1}, & h_3 &= h_1 + h_2 \\ e_4 &= e_2e_3 - e_3e_2, & e_{-4} &= e_{-3}e_{-2} - e_{-2}e_{-3}, & h_4 &= h_1 + 2h_2, \end{aligned} \quad (3.10)$$

where $h_1 = H_1, h_2 = \frac{1}{2}H_2$, $e_{\pm 1} = X_1^{\pm}$ and $e_{\pm 2} = \sqrt{[\frac{1}{2}]}X_2^{\pm}$. Up to a trivial automorphism, (3.10) agrees with the basis used in [36]. The reality structure is

$$\overline{e_1} = -e_{-1}, \quad \overline{e_2} = e_{-2}, \quad \overline{e_3} = -e_{-3}, \quad \overline{e_4} = -e_{-4}. \quad (3.11)$$

So $\{e_{\pm 2}, h_2\}$ is a $SU_{q^{\frac{1}{2}}}(2)$ algebra (but not coalgebra), and the other three $\{e_{\pm \beta}, h_{\beta}\}$ are noncompact $SO_{\bar{q}}(2, 1)$ algebras.

Now we can study q -deformed positive energy representations such as vector fields. As pointed out before, the representation theory is completely analogous to the classical case if q is not a root of unity, at least for finite-dimensional representations. In our case as well, it is easy to see that

$$V_{(E_0, 0)} \otimes V_5 = V_{(E_0, 1)} \oplus V_{(E_0+1, 0)} \oplus V_{(E_0-1, 0)} \quad (3.12)$$

as before for $E_0 \geq 2$, and the representation spaces are the same as classically. Then everything is as in section 3.1.2, however we will see below that none of these representations is unitary unless q is a root of unity.

In the following sections, we will show that for suitable roots of unity, there are unitary representations of $SO_q(2, 3)$ corresponding to all the classical ones mentioned above [56]. They are all finite-dimensional, and obtained from "compact" representations by a simple shift in energy. Moreover a BRST operator Q will arise naturally, for any spin. We start with the 2-dimensional case, which is technically simpler.

3.2.1 Unitary Representations of $SO_q(2, 1)$

In this section, we will use some results of [30] on $SU_q(2)$, where $2J$ equals H in our notation. $SO_q(2, 1)$ is defined by

$$\begin{aligned} [H, X^{\pm}] &= \pm 2X^{\pm}, & [X^+, X^-] &= [H]_q \\ \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(X^{\pm}) &= X^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X^{\pm}, \\ S(X^+) &= -qX^+, & S(X^-) &= -q^{-1}X^-, & S(H) &= -H \\ \epsilon(X^{\pm}) &= \epsilon(H) = 0 \end{aligned} \quad (3.13)$$

with the reality structure

$$\overline{H} = H, \quad \overline{X^+} = -X^- \quad (3.14)$$

as explained in section 3.1.1. Comparing with (1.15), this corresponds to the normalization $d = (\alpha, \alpha)/2 = 1$, but one can easily change to other normalizations by rescaling q , as in section 1.1.2.

The irreps of $U_q(su(2))$ at roots of unity are well – known [30], and we list some facts. As in section 1.2.3, for

$$q = e^{2\pi i n/m} \quad (3.15)$$

with positive relatively prime integers m, n let $M = m$ if m is odd, and $M = m/2$ if m is even. As explained in general, we can assume that

$$(X^\pm)^M = 0 \quad (3.16)$$

on all irreps (this excludes cyclic representations). Then all finite - dimensional irreps are highest weight (h.w.) representations with dimension $d \leq M$. There are two types of irreps:

- $V_{d,z} = \{e_m^j; \quad j = (d-1) + \frac{m}{2n}z, \quad m = j, j-2, \dots, -(d-1) + \frac{m}{2n}z\}$ with dimension d , for any $1 \leq d \leq M$ and $z \in \mathbb{Z}$, where $He_m^j = me_m^j$
- I_z^1 with dimension M and h.w. $(M-1) + \frac{m}{2n}z$, for $z \in \mathbb{C} \setminus \{\mathbb{Z} + \frac{2n}{m}r, 1 \leq r \leq M-1\}$.

Note that in the second type, $z \in \mathbb{Z}$ is allowed, in which case we will write $V_{M,z} \equiv I_z^1$ for convenience. We will concentrate on the $V_{d,z}$ – representations from now on. Furthermore, the fusion rules at roots of unity state that $V_{d,z} \otimes V_{d',z'}$ decomposes into $\oplus_{d''} V_{d'',z+z'} \oplus_p I_{z+z'}^p$ where I_z^p are the well - known reducible, but indecomposable representations of dimension $2M$, see figure 3.5 and [30].

Let us consider the invariant inner product (u, v) for $u, v \in V_{d,z}$, as defined in section 1.2.1, i.e. \overline{x} is the adjoint of $x \in \mathcal{U}$. If $(\ , \)$ is positive-definite, we have a unitary representation.

Proposition 3.2.1 *The representations $V_{d,z}$ are unitarizable w.r.t $SO_q(2, 1)$ if and only if*

$$(-1)^{z+1} \sin(2\pi nk/m) \sin(2\pi n(d-k)/m) > 0 \quad (3.17)$$

for all $k = 1, \dots, (d-1)$.

For $d - 1 < \frac{m}{2n}$, this holds precisely if z is odd. For $d - 1 \geq \frac{m}{2n}$, it holds for isolated values of d only, i.e. if it holds for d , then it (generally) does not hold for $d \pm 1, d \pm 2, \dots$

The representations $V_{d,z}$ are unitarizable w.r.t $SU_q(2)$ if z is even and $d - 1 < \frac{m}{2n}$.

Proof Let e_m^j be a basis of $V_{d,z}$ with h.w. j . After a straightforward calculation, invariance implies

$$\left((X^-)^k \cdot e_j^j, (X^-)^k \cdot e_j^j\right) = (-1)^k [k]! [j] [j - 1] \dots [j - k + 1] \left(e_j^j, e_j^j\right) \quad (3.18)$$

for $k = 1, \dots, (d - 1)$, where $[n]! = [1][2] \dots [n]$. Therefore we can have a positive definite inner product $(e_m^j, e_n^j) = \delta_{m,n}$ if and only if $a_k \equiv (-1)^k [k]! [j] [j - 1] \dots [j - k + 1]$ is a positive number for all $k = 1, \dots, (d - 1)$, in which case $e_{j-2k}^j = (a_k)^{-1/2} (X^-)^k \cdot e_j^j$.

Now $a_k = -[k][j - k + 1]a_{k-1}$, and

$$\begin{aligned} -[k][j - k + 1] &= -[k][d - k + \frac{m}{2n}z] = -[k][d - k]e^{i\pi z} \\ &= (-1)^{z+1} \sin(2\pi nk/m) \sin(2\pi n(d - k)/m) \frac{1}{\sin(2\pi n/m)^2}, \end{aligned} \quad (3.19)$$

since z is an integer. Then the Proposition follows. The compact case is known [30].

□

In particular, all of them are finite-dimensional, and clearly if q is not a root of unity, none of the representations are unitarizable.

We will be particularly interested in the case of (half)integer representations of type $V_{d,z}$ and $n = 1, m$ even, for reasons to be discussed below. Then $d - 1 < \frac{m}{2n} = M$ always holds, and the $V_{d,z}$ are unitarizable if and only if z is odd. These representations are centered around Mz , with dimension $\leq M$.

Let us compare this with the classical case. For the Anti-de Sitter group $SO(2, 1)$, H is nothing but the energy. At $q = 1$, the unitary irreps of $SO(2, 1)$ are lowest weight representations with lowest weight $j > 0$ resp. highest weight representations with highest weight $j < 0$. For any given such lowest resp. highest weight we can now find a *finite-dimensional* unitary representation with the same lowest resp. highest weight, provided M is large enough (we only consider (half)integer j here). These are unitary representations which for low energies look like the classical one-particle representations, but have an intrinsic high-energy cutoff if $q \neq 1$, which goes to infinity as $q \rightarrow 1$. The same will be true in the 4-dimensional case.

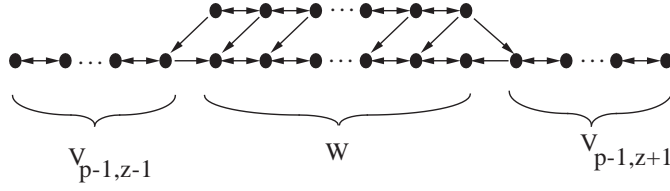


Figure 3.5: Indecomposable representation I_z^p

3.2.2 Tensor Product and Many-Particle Representations of $SO_q(2, 1)$

So far we only considered what could be called one-particle representations. Many-particle representations should be defined by some tensor product of 2 or more such irreps, which should be unitary as well and agree with the classical case at least for low energies.

Since \mathcal{U} is a Hopf algebra, there is a natural notion of a tensor product of two representations, given by the coproduct Δ . However, it is not unitary a priori. As mentioned above, the tensor product of two irreps of type $V_{d,z}$ is

$$V_{d,z} \otimes V_{d',z'} = \oplus_{d''} V_{d'',z+z'} \bigoplus_{p=r,r+2,\dots}^{d+d'-M} I_{z+z'}^p \quad (3.20)$$

where $r = 1$ if $d+d'-M$ is odd or else $r = 2$, and I_z^p is a indecomposable representation of dimension $2M$ whose structure is shown in figure 3.5. The arrows indicate the rising resp. lowering operators.

In the case of $SU_q(2)$, one defines a truncated tensor product $\hat{\otimes}$ in the context of CFT by omitting all I_z^p representations [40]. Then the remaining reps are unitary w.r.t. $SU_q(2)$; see [40].

This is not the right thing to do for $SO_q(2, 1)$. Let $n = 1$ and m even, and consider e.g. $V_{M-1,1} \otimes V_{M-1,1}$. Both factors have lowest energy $H = 2$, and the tensor product of the two corresponding *classical* representations is the sum of representations with lowest weights $4, 6, 8, \dots$. In our case, these weights are in the I_z^p representations, while the $V_{d'',z''}$ have $H \geq M \rightarrow \infty$ and are not unitarizable. So we have to keep the I_z^p 's and throw away the $V_{d'',z''}$'s in (3.20). A priori however, the I_z^p 's are not unitarizable, either. To get a unitary tensor product, note that as a vector space,

$$I_z^p = V_{p-1,z-1} \oplus W \oplus V_{p-1,z+1} \quad (3.21)$$

where

$$W = V_{M-p+1,z} \oplus V_{M-p+1,z} \quad (3.22)$$

as vector space. Now $(X^+)^{p-1} \cdot e_l$ is a lowest weight state where e_l is the lowest weight vector of I_p^z , and similarly $(X^-)^{p-1} \cdot e_h$ is a highest weight state with e_h being the highest weight vector of I_p^z (see figure 3.5). It is therefore consistent to consider the submodule of I_p^z generated by e_l , and factor out its submodule generated by $(X^+)^{p-1} \cdot e_l$; the result is an irreducible representation equivalent to $V_{p-1,z-1}$ realized on the left summand in (3.21). Similarly, one could consider the submodule of I_p^z generated by e_h , factor out its submodule generated by $(X^-)^{p-1} \cdot e_h$, and obtain an irreducible representation equivalent to $V_{p-1,z+1}$. In short, one can just "delete" W in (3.21). These two V -type representations are unitarizable provided $n = 1$ and m is even, and one can either keep both (notice the similarity with band structures in solid-state physics), or for simplicity keep the low-energy part only, in view of the physical application we have in mind. We therefore define a truncated tensor product as

Definition 3.2.2 For $n = 1$ and even m ,

$$V_{d,z} \tilde{\otimes} V_{d',z'} \equiv \bigoplus_{\tilde{d}=r, r+2, \dots}^{d+d'-M} V_{\tilde{d}, z+z'-1} \quad (3.23)$$

This can be stated as follows: Notice that any representation naturally decomposes as a vector space into sums of $V_{d,z}$'s, cp. (3.22); the definition of $\tilde{\otimes}$ simply means that only the smallest value of z in this decomposition is kept, which is the submodule of irreps with lowest weights $\leq \frac{m}{2n}(z + z' - 1)$. With this in mind, it is obvious that $\tilde{\otimes}$ is associative: both in $(V_1 \tilde{\otimes} V_2) \tilde{\otimes} V_3$ and in $V_1 \tilde{\otimes} (V_2 \tilde{\otimes} V_3)$, the result is simply the V 's with minimal z , which is the *same* space, because the ordinary tensor product is associative and Δ is coassociative. This is in contrast with the "ordinary" truncated tensor product $\hat{\otimes}$ [40]. Of course, one could give a similar definition for negative-energy representations.

This will be generalized to the 4-dimensional case, and in a later section, we will give a conjecture on a elegant definition of a completely reducible tensor product using a BRST operator.

$V_{d,z} \tilde{\otimes} V_{d',z'}$ is unitarizable if all the V 's on the rhs of (3.23) are unitarizable. This is certainly true if $n = 1$ and m is even. In all other cases, there are no terms on the rhs of (3.23) if the factors on the lhs are unitarizable, since no I_z^p -type representations

are generated (they are too large). This is the reason why we concentrate on this case, and furthermore on $z = z' = 1$ which corresponds to low-energy representations. Then $\tilde{\otimes}$ defines a two-particle Hilbert space with the correct classical limit. So

Proposition 3.2.3 $\tilde{\otimes}$ is associative, and $V_{d,1}\tilde{\otimes}V_{d',1}$ is unitarizable.

How the inner product can be induced from the single-particle Hilbert spaces will be explained in section 4.2.

Before discussing $SO_q(2, 3)$, we will consider the compact case.

3.2.3 Unitary Representations of $SO_q(5)$

Again $q = e^{2\pi i n/m}$. As explained in sections 1.2.1 and 1.2.3, the irreducible h.w. representations $L(\lambda)$ with highest weight λ can be obtained from the corresponding Verma module $M(\lambda)$ by factoring out its maximal submodule. The latter is precisely the null spaces w.r.t. its invariant inner product, and this is what we have to determine first.

The following discussion until the paragraph before Definition 3.2.6 is technical and may be skipped upon first reading. As in section 1.2, $Q = \sum \mathbb{Z}\alpha_i$ is the root lattice, $Q^+ = \sum \mathbb{Z}_+\alpha_i$, and

$$\lambda \succ \mu \quad \text{if} \quad \lambda - \mu \in Q^+. \quad (3.24)$$

For $\eta \in Q$, denote [13]

$$\text{Par}(\eta) = \{\underline{k} \in \mathbb{Z}_+^N; \quad \sum k_i \beta_i = \eta\}. \quad (3.25)$$

Let $M(\lambda)_\eta$ be the weight space with weight $\lambda - \eta$ in $M(\lambda)$. Then its dimension is given by $|\text{Par}(\eta)|$. If $M(\lambda)$ contains a h.w. vector with weight σ , then the multiplicity of the weight space $(M(\lambda)/M(\sigma))_\eta$ is given by $|\text{Par}(\eta)| - |\text{Par}(\eta + \sigma - \lambda)|$, and so on.

The character of a representation $V(\lambda)$ with maximal weight λ is the function on weight space defined by

$$\text{ch } V(\lambda) = e^\lambda \sum_{\eta \in Q^+} \dim V(\lambda)_\eta e^{-\eta}, \quad (3.26)$$

where again $V(\lambda)_\eta$ is the weight space of $V(\lambda)$ at weight $\lambda - \eta$, and $e^{\lambda-\eta}(\mu) \equiv e^{(\lambda-\eta, \mu)}$. The characters of inequivalent highest weight irreps are linearly independent;

remember that they are all finite-dimensional at roots of unity. The sum makes sense even for Verma modules and agrees with the classical result,

$$\text{ch } M(\lambda) = e^\lambda \sum_{\eta \in Q^+} |\text{Par}(\eta)| e^{-\eta}, \quad (3.27)$$

see [26].

In general, the structure of Verma modules is complicated and it is not always enough to know all highest weight vectors, cp. [26]. The proper technical tool to describe the structure of a Verma module is its *composition series*, or Jordan–Hölder series. For any module M with a maximal weight, consider a sequence of nested submodules $\dots \subset W_2 \subset W_1 \subset W_0 = M$ such that W_k/W_{k+1} is irreducible, and thus $W_k/W_{k+1} \cong L(\mu_k)$ for some μ_k ; this is called a Jordan–Hölder series (it is infinite for roots of unity, but this is not a problem for the following arguments). It can be constructed inductively by fixing a maximal submodule of the W_k (e.g. by factoring out inductively all but one highest weight submodules of W_k , the sum of which is a possible W_{k+1}). There are many ways to construct a Jordan–Hölder series, but for all of them we obviously have $\text{ch } M = \sum \text{ch } (W_k/W_{k+1}) = \sum \text{ch } L(\mu_k)$. Since the characters of irreps are linearly independent, this decomposition of $\text{ch } M$ is unique, and so are the subquotients $L(\mu_k)$. We want to determine these $L(\mu_k)$.

The main tool to find them will be a remarkable formula by De Concini and Kac for $\det(M(\lambda)_\eta)$, the determinant of the invariant inner product matrix of $M(\lambda)_\eta$ in a P.B.W. basis, for arbitrary highest weight λ . For $|q| = 1$, their result is as follows [13]:

$$\det(M(\lambda)_\eta) = \prod_{\beta \in R^+} \prod_{k \in \mathbb{N}} \left([k]_{d_\beta} \frac{q^{(\lambda + \rho - k\beta/2, \beta)} - q^{-(\lambda + \rho - k\beta/2, \beta)}}{q^{d_\beta} - q^{-d_\beta}} \right)^{|\text{Par}(\eta - k\beta)|} \quad (3.28)$$

where R^+ denotes the positive roots, $d_\beta = (\beta, \beta)/2$, and $k = k_\beta$ really.

To get some insight, notice first of all that due to $|\text{Par}(\eta - k\beta)|$ in the exponent, the product is finite. Now for some positive root β , let k_β be the smallest integer such that $D(\lambda)_{k_\beta, \beta} \equiv \left([k_\beta]_{d_\beta} \frac{q^{(\lambda + \rho - k_\beta\beta/2, \beta)} - q^{-(\lambda + \rho - k_\beta\beta/2, \beta)}}{q^{d_\beta} - q^{-d_\beta}} \right) = 0$ (assuming such a k_β exists) and consider the weight space at weight $\lambda - k_\beta\beta$, i.e. $\eta_\beta = k_\beta\beta$. Then $|\text{Par}(\eta_\beta - k_\beta\beta)| = 1$ and $\det(M(\lambda)_{\eta_\beta})$ is zero, so there is a h.w. vector w_β with weight $\lambda - \eta_\beta$ (assuming that there is no other with weight $\succ (\lambda - \eta_\beta)$). It generates a submodule which is again a Verma module (because \mathcal{U} does not have zero divisors [13]), with dimension $|\text{Par}(\eta - k_\beta\beta)|$ at weight $\lambda - \eta$. This is the origin of the exponent. However the submodules generated by the ω_{β_i} are not independent, i.e. they contain common h.w.

vectors, and there might be other h.w. vectors at different weights. Nevertheless, we will see that all the highest weights μ_k of the composition series of $M(\lambda)$ are precisely obtained in this way. This "strong linkage principle" will be formulated carefully below. The corresponding statement in the classical case is well-known [26]. While it is not a new insight for the quantum case either [12, 2], it seems that no explicit proof applicable to our purpose has been given (the results in [2] apply only to certain odd roots of unity, and we will see that in fact the even ones are most interesting here), and we will provide one, adapting arguments in [26].

To make the structure more transparent, let \mathbb{N}_β^T be the set of positive integers k with $[k]_\beta = 0$, and \mathbb{N}_β^R the positive integers k such that $(\lambda + \rho - \frac{k}{2}\beta, \beta) \in \frac{m}{2n}\mathbb{Z}$. Then

$$D(\lambda)_{k,\beta} = 0 \Leftrightarrow k \in \mathbb{N}_\beta^T \quad \text{or} \quad k \in \mathbb{N}_\beta^R. \quad (3.29)$$

The second condition is $k = 2 \frac{(\lambda+\rho, \beta)}{(\beta, \beta)} + \frac{m}{2n} \frac{2}{(\beta, \beta)} \mathbb{Z}$, which means that

$$\lambda - k\beta = \sigma_{\beta,l}(\lambda) \quad (3.30)$$

where $\sigma_{\beta,l}(\lambda)$ is the reflection of λ by a plane perpendicular to β through $-\rho + \frac{m}{4nd_\beta}l\beta$, for some integer l . For general l , $\sigma_{\beta,l}(\lambda) \notin \lambda + Q$; but k should be an integer, so it is natural to define the (*modified*) *affine Weyl group* \mathcal{W}_λ of reflections in weight space to be generated by those σ_{β_i, l_i} which map λ into $\lambda + Q$, cp. section 1.2.3. For $q = e^{2\pi i n/m}$, two such allowed reflection planes $\perp \beta_i$ will differ by multiples of $\frac{1}{2}M_{(i)}\beta_i$; in the case of $SO_q(5)$, $M_{(2,3)} = m$ and $M_{(1,4)} = m$ resp. $m/2$ if m is odd resp. even. Thus \mathcal{W}_λ is generated by all reflections by these planes. Alternatively, it is generated by the usual Weyl group with a suitable reflection center, and translations by $M_{(i)}\beta_i$, which correspond to $\mathbb{N}_{\beta_i}^T$.

Now the *strong linkage principle* states the following:

Theorem 3.2.4 *$L(\mu)$ is a composition factor of the Verma module $M(\lambda)$ if and only if μ is strongly linked to λ , i.e. if there is a descendant sequence of weights related by the affine Weyl group as*

$$\lambda \succ \lambda_i = \sigma_{\beta_i, l_i}(\lambda) \succ \dots \succ \lambda_{kj\dots i} = \sigma_{\beta_k, l_k}(\lambda_{j\dots i}) = \mu \quad (3.31)$$

Proof The main tool to show this is the formula (3.28). To make use of it, deform λ to $\lambda' = \lambda + h\rho$ for² $h \in \mathbb{C}$, and q to $q' = qe^{i\pi h}$, so that $D(\lambda')_{k\beta,\beta} \neq 0$. Consider the inner product matrix $(a, b)_{\lambda'}$ for a, b being P.B.W. basis vectors of $M(\lambda')$; here h is treated as a formal variable, i.e. no complex conjugation is implied by the "sesquilinear" form (this is customary in the mathematical literature). Then $(a, b)_{\lambda'}$ is hermitian if $h \in \mathbb{R}$, and (3.28) holds for any $h \in \mathbb{C}$.

Although the $M(\lambda')$ strictly speaking depend on λ' , we can identify them for different h via the P.B.W. basis. In this sense, the action of X_i^\pm is analytic in h since it only depends on the commutation relations of the X_β^\pm , cp. [13], and so is $(a, b)_{\lambda'}$. According to a theorem ([29], chapter 2, theorem 1.10) for analytic matrices which are normal for real h , its eigenvalues e_α are analytic, and there exist analytic projectors P_{e_α} on the eigenspaces V_{e_α} which span the entire vectorspace (except possibly at isolated points where some eigenvalues coincide; for $h \in \mathbb{R}$ however, the generic eigenspaces are orthogonal and therefore remain independent even at such points). These projectors provide an analytic basis of eigenvectors of $(a, b)_{\lambda'}$ near λ . We can now define

$$V_k \equiv \bigoplus_{e_\alpha \propto h^k} V_{e_\alpha}, \quad (3.32)$$

i.e. the sum of the eigenspaces with eigenvalues e_α with a zero of order k (precisely) at $h = 0$. Of course, $V_k \perp V_{k'}$ for $k \neq k'$. The V_k span the entire space, they have an analytic basis as discussed, and have the following properties:

Lemma 3.2.5 1) $(v_k, v)_{\lambda'} = o(h^k)$ for $v_k \in V_k$ and any (analytic) $v \in M(\lambda')$.

2) $X_i^\pm v_k = \sum_{l \geq k} a_l v_l + \sum_{l=1}^k h^l b_l v_{k-l}$ for $v_l \in V_l$ and a_l, b_l analytic. In particular at $h = 0$,

$$M^k \equiv \bigoplus_{n \geq k} V_n \quad (3.33)$$

is invariant.

Proof

- 1) Decomposing v according to $\bigoplus_l V_l$, only the (analytic) component in V_k contributes in (v_k, v) , with a factor h^k by the definition of V_k ($o(h^k)$ means at least k factors of h).

²on complex weights, see [13] below Prop. 1.9.

- 2) Decompose $X_i^\pm v_k = \sum_{e_\alpha} a_{e_\alpha} v_{e_\alpha}$ with analytic coefficients a_{e_α} , corresponding to the eigenvalue e_α . For any v_{e_α} appearing on the rhs, consider $(v_{e_\alpha}, X_i^\pm v_k) = a_{e_\alpha}(v_{e_\alpha}, v_{e_\alpha}) = c a_{e_\alpha} e_\alpha$ with $c \neq 0$ at $h = 0$ (v_{e_α} might not be normalized). But the lhs is $(X_i^\mp v_{e_\alpha}, v_k) = o(h^k)$ as shown above. Therefore $a_{e_\alpha} e_\alpha = o(h^k)$, which implies 2).

□

In particular, the quotient $M(\lambda)/_{M^1}$ is irreducible and isomorphic to $L(\lambda)$. (The sequence of submodules $\dots \subset M^2 \subset M^1 \subset M(\lambda)$ is similar to the Jantzen filtration [26].)

By the definition of V_n resp. M^k , we have

$$\text{ord}(\det(M(\lambda)_\eta)) = \sum_{k \geq 1} \dim M_{\lambda-\eta}^k \quad (3.34)$$

where $M_{\lambda-\eta}^k$ is the weight space of M^k at weight $\lambda - \eta$, and $\text{ord}(\det(M(\lambda)_\eta))$ is the order of the zero of $\det(M(\lambda)_\eta)$ as a function of h , i.e. the maximal power of h it contains. Now from (3.28) and the above definition of $\mathbb{N}_\beta^{T,R}$, it follows

$$\begin{aligned} \sum_{k \geq 1} \text{ch } M^k &= e^\lambda \sum_{\eta \in Q^+} \left(\sum_{k \geq 1} \dim M_{\lambda-\eta}^k \right) e^{-\eta} \\ &= e^\lambda \sum_{\eta \in Q^+} \text{ord}(\det(M(\lambda)_\eta)) e^{-\eta} \\ &= \sum_{\beta \in \mathcal{R}^+} \left(\sum_{n \in \mathbb{N}_\beta^T} + \sum_{n \in \mathbb{N}_\beta^R} \right) e^\lambda \sum_{\eta \in Q^+} |\text{Par}(\eta - n\beta)| e^{-\eta} \\ &= \sum_{\beta \in \mathcal{R}^+} \left(\sum_{n \in \mathbb{N}_\beta^T} + \sum_{n \in \mathbb{N}_\beta^R} \right) \text{ch } M(\lambda - n\beta) \end{aligned} \quad (3.35)$$

where we used (3.27).

Now we can now prove (3.2.4) inductively. Both the left and the right side of (3.35) can be decomposed into a sum of characters of highest weight irreps, according to their Jordan–Hölder series. These characters are linearly independent. Suppose that $L(\lambda - \eta)$ is a composition factor of $M(\lambda)$. Then the corresponding character is contained in the lhs of (3.35), since $M(\lambda)/_{M^1}$ is irreducible. Therefore it is also contained in one of the $\text{ch } M(\lambda - n\beta)$ on the rhs. Therefore $L(\lambda - \eta)$ is a composition factor of one of these $M(\lambda - n\beta)$, and by the induction assumption we obtain that $\mu \equiv \lambda - \eta$ is strongly linked to λ as in (3.31).

Conversely, assume that μ satisfies (3.31). By the induction assumption, there exists a $n \in \mathbb{N}_\beta^T \cup \mathbb{N}_\beta^R$ such that $L(\mu)$ is a subquotient of $M(\lambda - n\beta)$. Then (3.35) shows that $L(\mu)$ is a subquotient of $M(\lambda)$. \square

Obviously this applies to other quantum groups as well. In particular, we see again that for $q = e^{2\pi i n/m}$, all $(X_i^-)^{M(i)} w_\lambda$ are h.w. vectors, and factored out in an irrep.

With these tools, we are now ready to study irreps and determine which ones are unitarizable, i.e. for which the inner product is positive definite. As mentioned before, there exist remarkable nontrivial one-dimensional representations w_{λ_0} with weights $\lambda_0 = \sum \frac{m}{2n} k_i \alpha_i$ for integers k_i . By tensoring any representation with w_{λ_0} , one obtains another representation with identical structure, but all weights shifted by λ_0 . We will see below that by such a shift, representations which are unitarizable w.r.t. $SO_q(2,3)$ are in one-to-one correspondence with representations which are unitarizable w.r.t. $SO_q(5)$. It is therefore enough to consider highest weights in the following domain:

Definition 3.2.6 *A weight $\lambda = E_0\beta_3 + s\beta_2$ is called basic if*

$$0 \leq (\lambda, \beta_3) = E_0 < \frac{m}{2n}, \quad 0 \leq (\lambda, \beta_4) = (E_0 + s) < \frac{m}{2n}. \quad (3.36)$$

In particular, $\lambda \succ 0$. It is compact if in addition it is integral (i.e. $(\lambda, \beta_i) \in \mathbb{Z}d_i$),

$$s \geq 0 \quad \text{and} \quad (\lambda, \beta_1) \geq 0. \quad (3.37)$$

An irrep with compact h.w. will be called compact.

The region of basic weights is drawn in figure 3.6, together with the lattice of w_{λ_0} 's. The compact representations are centered around 0, and the (quantum) Weyl group [33] acts on them, as classically (it is easy to see that the action of the quantum Weyl group resp. braid group on the compact representations is well defined at roots of unity as well).

A representation with basic highest weight can be unitarizable w.r.t. $SO_q(5)$ (with conjugation $\overline{(\cdot)}^c$) only if all the $SU(2)$'s are unitarizable. For compact λ , all the $SU_q(2)$'s are indeed unitarizable according to section 3.2.2, using $M_{(2,3)} = m$ and $M_{(1,4)} = m$ resp. $m/2$ if m is odd resp. even. This alone however is not enough to show that they are unitarizable w.r.t. to the full group.

Although it may seem surprising, there are actually unitary representations with nonintegral basic highest weight, namely for

$$\lambda = \frac{m-1}{2}\beta_3 \quad \text{and} \quad \lambda = \left(\frac{m}{2} - 1\right)\beta_3 + \frac{1}{2}\beta_2 \quad (3.38)$$

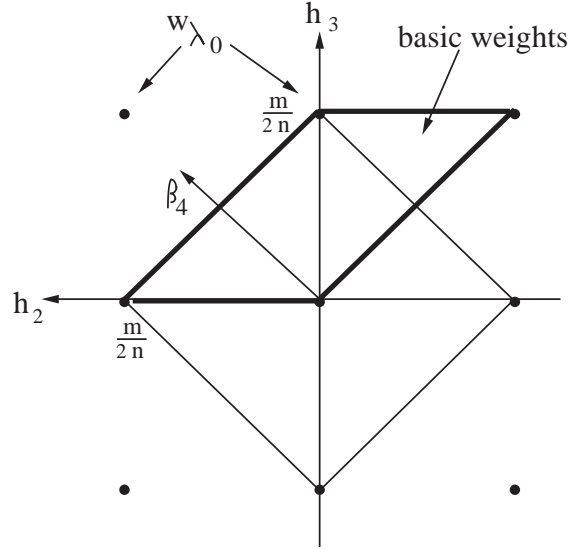


Figure 3.6: Envelope of compact representations, basic weights and the lattice of w_{λ_0}

for $n = 1$ and m even. It follows from theorem (3.2.4) that there is a h.w. vector at $\lambda - 2\beta_3$ resp. $\lambda - \beta_3$, and all the multiplicities turn out to be one in the irreps. Thus all $SU_q(2)$ modules in β_1, β_4 direction have maximal length $M_{(1)} = m/2$, from which it follows that they are unitarizable. The structure is that of shifted Dirac singletons which were already studied in [12], and we will come back to them.

It appears that all other irreps must have integral highest weight in order to be unitarizable w.r.t. $SO_q(5)$. Furthermore, if the highest weight is not compact, some of the $SU_q(2)$'s will not be unitarizable. On the other hand, all irreps with compact highest weight are indeed unitarizable:

Theorem 3.2.7 *The structure of the irreps $V(\lambda)$ with compact highest weight λ is the same as classically except in the cases*

- a) $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ for $s \geq 1$ and $\frac{m}{2n}$ integer, where one additional highest weight state at weight $\lambda - \beta_4$ appears and no others, and
- b) $\lambda = \frac{m-1}{2}\beta_3$ and $\lambda = (\frac{m}{2} - 1)\beta_3 + \frac{1}{2}\beta_2$ for $n = 1$ and m odd, where one additional highest weight state at weight $\lambda - 2\beta_3$ resp. $\lambda - \beta_3$ appears and no others,

which are factored out in the irrep. They are unitarizable w.r.t. $SO_q(5)$ (with conjugation θ^).*

The irreps with nonintegral highest weights (3.38) as discussed above are unitarizable as well.

Proof The statements on the structure follow easily from theorem 3.2.4.

To show that these irreps are unitarizable, consider the compact representation with highest weight λ before factoring out the additional h.w. state, so that the space is the same as classically. For $q = 1$, they are known to be unitarizable, so the inner product is positive definite. Consider the eigenvalues of the inner product matrix of $(\ , \)_q$ as q goes from 1 to $e^{2\pi i n/m}$ along the unit circle. The only way an eigenvalue could become negative is that it is zero for some $q' \neq q$. This can only happen if q' is a root of unity, $q' = e^{2i\pi n'/m'}$ with $n'/m' < n/m$. But then the "non-classical" reflection planes of \mathcal{W}_λ are further away from the origin and are relevant only in the case $\lambda = \frac{m-1}{2}\beta_3$ for $n = 1$ and m odd; but as pointed out above, no additional eigenvector appears in this case for $q' \neq q$.

Thus the eigenvalues might only become zero at q . This happens precisely if a new h.w. vector appears, i.e. in the cases listed. Since there is no null vector in the remaining irrep, all its eigenvalues are positive by continuity. \square

So far all results were stated for h.w. modules; of course the analogous statements for lowest weight modules are true as well. All the $V(\lambda)$ in the above theorem have lowest weight $-\lambda$.

3.2.4 Unitary Representations of $SO_q(2, 3)$

In this section, we will finally see that there are finite-dimensional, unitary positive-energy irreps of $SO_q(2, 3)$ corresponding to all the classical unitary representations discussed in section 3.1.2, for suitable roots of unity q . At low energies, their structure is the same as classically including the appearance of "pure gauge" subspaces in the massless case, for spin ≥ 1 . Again, these "pure gauge states" can be factored out to obtain the physical, unitary representations. At high energies, there is an intrinsic cutoff.

These lowest weight representations can be obtained from the compact ones by a shift, as indicated in section 1.2.3: if $V(\lambda)$ is a compact h.w. representation, then

$$V_{(\mu)} \equiv V(\lambda) \otimes \omega \quad (3.39)$$

with $\omega \equiv w_{\lambda_0}$, $\lambda_0 = \frac{m}{2n}\beta_3$ has lowest weight $\mu = -\lambda + \lambda_0 \equiv E_0\beta_3 - s\beta_2 \equiv (E_0, s)$. It is a positive-energy representation, i.e. the eigenvalues of h_3 are positive.

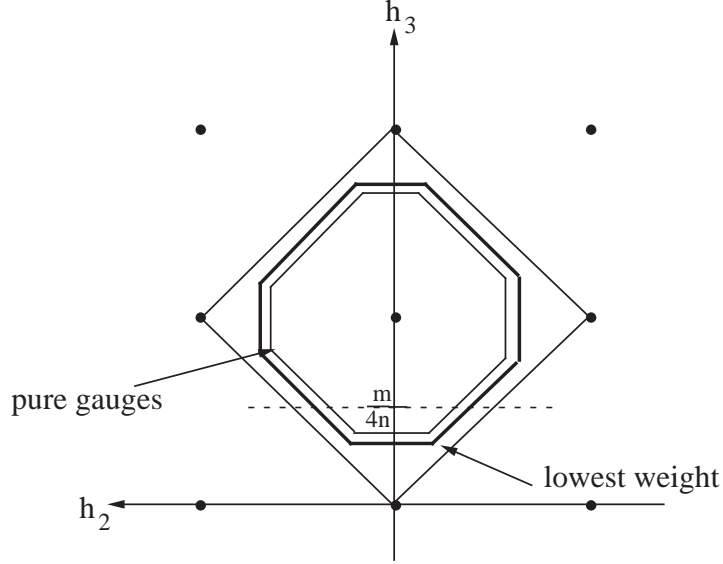


Figure 3.7: Physical representation with subspace of pure gauges (for $\frac{m}{2n}$ integer), schematically. For $h_3 \leq \frac{m}{4n}$, the structure is the same as for $q = 1$.

For $\frac{m}{2n}$ integer, $V_{(\mu)}$ corresponds precisely to the classical positive-energy representation with the same lowest weight. Again, E_0 is the rest energy and s the spin. For $h_3 \leq m/4n$, the structure is the same as classically, see figure 3.7. The irreps with nonintegral highest weights (3.38) correspond upon this shift to the Dirac singleton representations "Rac" with lowest weight $\mu = (1/2, 0)$ and "Di" with $\mu = (1, 1/2)$, as discussed in [12].

If $\frac{m}{2n}$ is not integer, the weights of shifted compact representations are not integral. For $n = 1$ and m odd, the irreps in b) of theorem (3.2.7) now correspond to the singletons, again in agreement with [12]. We will see however that this case does not lead to an interesting tensor product.

For $\frac{m}{2n}$ integer, the cases $\mu = (s + 1, s)$ for $s \geq 1$ will be called "massless" for the same reasons as in section 3.1.2. E_0 is the smallest possible rest energy for a unitarizable representation with given s (see below), and an additional lowest weight state with $E'_0 = E_0 + 1$ and $s' = s - 1$ appears as classically, which generates a null-subspace of "pure gauge" states. But now, all these representations are finite-dimensional.

This motivates the following

Definition 3.2.8 A lowest weight irrep $V_{(\mu)}$ with lowest weight $\mu = (E_0, s) \equiv E_0\beta_3 -$

$s\beta_2$ (resp. μ itself) is called physical if it is unitarizable w.r.t. $SO_q(2, 3)$ (with conjugation as in (3.2) ff.).

For $n = 1$, $V_{(\mu)}$ is called Di if $\mu = (1, 1/2)$ and Rac if $\mu = (1/2, 0)$.

For $\frac{m}{2n}$ integer, $V_{(\mu)}$ is called massless if $\mu = (s + 1, s)$ for $s \in \frac{1}{2}\mathbb{Z}$ and $s \geq 1$.

Theorem 3.2.9 *A lowest weight irrep $V_{(\mu)}$ is physical precisely if the shifted irrep with lowest weight $\mu - \frac{m}{2n}\beta_3$ is unitarizable w.r.t. $SO_q(5)$.*

All $V_{(\mu)}$ where $-(\mu - \frac{m}{2n}\beta_3)$ is compact are physical, in particular the massless irreps, as well as the singletons Di and Rac. For $h_3 \leq \frac{m}{4n}$, they are obtained from a (lowest weight) Verma module by factoring out the submodule with lowest weight $(E_0, -(s + 1))$ only, except for the massless case, where an additional lowest weight state with weight $(E_0 + 1, s - 1)$ appears, and for the Di resp. Rac, where an additional lowest weight state with weight $(E_0 + 1, s)$ resp. $(E_0 + 2, s)$ appears. This is the same as classically, see figure 3.7.

For the singletons, this was already shown in [12].

Proof As mentioned before, we can write every vector in such a representation uniquely as $a \cdot \omega$, where a belongs to a unitarizable irrep of $SO_q(5)$. Consider the inner product

$$\langle a \cdot \omega, b \cdot \omega \rangle \equiv (a, b), \quad (3.40)$$

where (a, b) is the hermitian inner product on the compact (shifted) representation. Then

$$\begin{aligned} \langle a \cdot \omega, e_1(b \cdot \omega) \rangle &= \langle a \cdot \omega, (e_1 \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_1)b \otimes \omega \rangle \\ &= q^{h_1/2}|_{\omega}(a, e_1 b) = i(a, e_1 b) \end{aligned} \quad (3.41)$$

using $h_1|_{\omega} = \frac{m}{2n}$. Similarly,

$$\begin{aligned} \langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle &= \langle (e_{-1} \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_{-1})a \otimes \omega, b \otimes \omega \rangle \\ &= q^{-h_1/2}|_{\omega}(e_{-1}a, b) = -i(e_{-1}a, b) \end{aligned} \quad (3.42)$$

because $\langle \cdot, \cdot \rangle$ is antilinear in the first argument and linear in the second. Therefore

$$\langle a \cdot \omega, e_1(b \cdot \omega) \rangle = -\langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle. \quad (3.43)$$

Similarly $\langle a \cdot \omega, e_2(b \cdot \omega) \rangle = \langle e_{-2}(a \cdot \omega), b \cdot \omega \rangle$. This shows that $\langle \cdot, \cdot \rangle$ is hermitian w.r.t. \bar{x} , and positive definite because (\cdot, \cdot) is positive definite by definition. Theorem 3.2.7 now completes the proof. \square

As a consistency check, one can see again from section 3.2.1 that all the $SO_{\tilde{q}}(2, 1)$ resp. $SU_{\tilde{q}}(2)$ subgroups are unitarizable in these representations, but this is not enough to show unitarizability for the full group. Note that as $m \rightarrow \infty$ for $n = 1$, one obtains the classical one-particle representations for given s, E_0 . We have therefore also proved the unitarizability at $q = 1$ for (half)integer spin, which appears to be non-trivial in itself [22]. Furthermore, *all representations obtained from the above by shifting E_0 or s by a multiple of $\frac{m}{n}$ are unitarizable as well.* One obtains in weight space a cell-like structure of representations which are unitarizable w.r.t. $SO_q(2, 3)$ resp. $SO_q(5)$.

3.2.5 Tensor Product and Many-Particle Representations of $SO_q(2, 3)$

Finally we want to consider many-particle representations, i.e. find a tensor product such that the tensor product of unitary representations is unitarizable, as in section 3.2.2. The idea is the same as there, the tensor product of 2 such representations will be a direct sum of representations, and we only keep appropriate physical lowest-weight subspaces. To make this more precise, consider two physical irreps $V_{(\mu)}$ and $V_{(\mu')}$ as in Definition (3.2.8). For a basis $\{u_{\lambda'}\}$ of physical lowest weight states in $V_{(\mu)} \otimes V_{(\mu')}$, consider the linear span $\langle \mathcal{U}u_{\lambda'} \rangle$ of its lowest-weight submodules, and let $Q_{\mu, \mu'}$ be the quotient of it after factoring out all proper submodules of the $\mathcal{U}u_{\lambda'}$. Let $\{u_{\lambda''}\}$ be a basis of lowest weight states of $Q_{\mu, \mu'}$. Then $Q_{\mu, \mu'} = \oplus V_{(\lambda'')}$ where $V_{(\lambda'')}$ are the corresponding (physical) irreducible lowest weight modules, i.e. $Q_{\mu, \mu'}$ is completely reducible. Therefore the following definition makes sense:

Definition 3.2.10 *In the above situation, let $\{u_{\lambda''}\}$ be a basis of physical lowest-weight states of $Q_{\mu, \mu'}$, and $V_{(\lambda'')}$ be the corresponding physical lowest weight irreps. Then define*

$$V_{(\mu)} \tilde{\otimes} V_{(\mu')} \equiv \bigoplus_{\lambda''} V_{(\lambda'')} \quad (3.44)$$

Notice that if $\frac{m}{2n}$ is not integer, then the physical states have non-integral weights, and the full tensor product of two physical irreps $V_{(\mu)} \otimes V_{(\mu')}$ does not contain any physical lowest weights. Therefore $V_{(\mu)} \tilde{\otimes} V_{(\mu')}$ is zero, and there seems to be no reasonable way to get around this.

Again as in section 3.2.2, one might also include a second "band" of high-energy states. Now

Theorem 3.2.11 *If all weights μ, μ', \dots involved are integral, then $\tilde{\otimes}$ is associative, and $V_{(\mu)} \tilde{\otimes} V_{(\mu')}$ is unitarizable w.r.t. $SO_q(2, 3)$.*

Proof First, notice that the λ'' are all integral and none of them gives rise to a massless representation or a singleton. So by the strong linkage principle, none of the $\mathcal{U}u_{\lambda'}$ can contain a physical lowest-weight submodule. Also, lowest weight states for generic q cannot disappear at roots of unity. Therefore $Q_{\mu, \mu'}$ contains all the physical lowest-weight states of the full tensor product. Furthermore, no physical lowest-weight states are contained in (discarded states) \otimes (any states). Then associativity follows from associativity of the full \otimes , and the structure is the same as classically for energies $h_3 \leq \frac{m}{4n}$ (observe that \otimes contains no massless representations, so classically inequivalent physical representations cannot recombine into indecomposable ones). \square

In particular, none of the low-energy states have been discarded. Therefore our definition is physically sensible, and the case of $q = e^{2\pi i/m}$ with m even appears to be most interesting physically.

The highest ("cosmological") energies available in this "low-energy band" are of order $E_{max} = \frac{1}{hR}$ in appropriate units, where R is the radius of AdS space, and $h = 1/m$. This is by a factor $\frac{1}{\sqrt{h}}$ larger than the energy scale $L_0 \approx \frac{1}{\sqrt{hR}}$ where the geometry becomes noncommutative, see section 2.3.2. From a QFT point of view, the latter should be the interesting scale. Thus the hierarchy from the curvature radius R to the geometrical scale L_0 is the same as between L_0 and $1/E_{max}$. This hierarchy has to be large in order to have a large number of physical states available. In any case, this shows that such a theory (imaginary, so far) could in principle accomodate large systems, with an interesting relation between "cosmological" scales and a geometrical scale (even though we do not advertise this as a cosmologically interesting model).

3.2.6 Massless Particles, Indecomposable Representations and BRST for $SO_q(2, 3)$

Let $n = 1$ and $m = 2M$, i.e. $q = e^{i\pi/M}$. $V_{(E_0, 0)}$ will again be called "scalar field", $V_{(E_0, 1)}$ "vector field", and so on. $V_{(E_0, s)}$ has a subspace of pure gauges in the "massless" case, otherwise they are irreducible. They are unitarizable w.r.t. $SO_q(2, 3)$ after factoring out the pure gauge states.

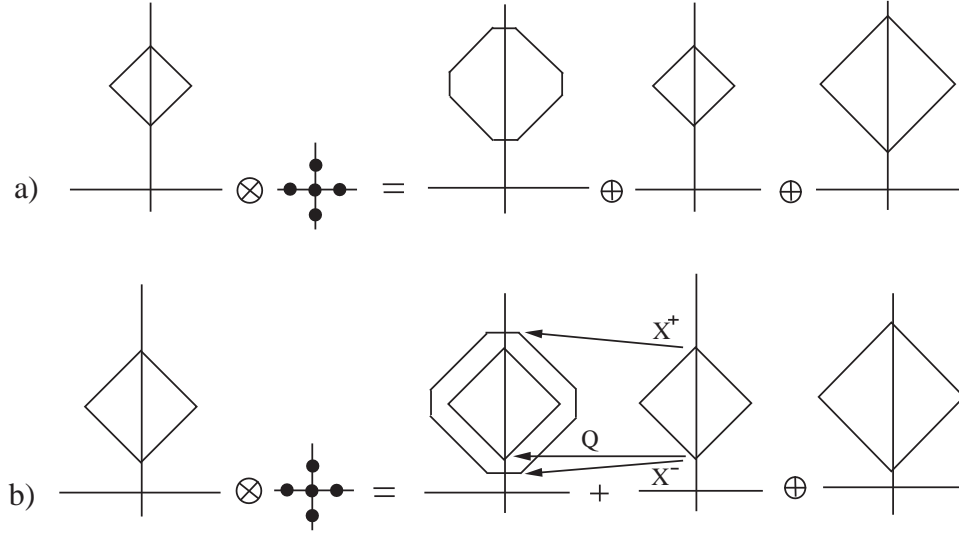


Figure 3.8: $V(\lambda) \otimes V_5$ for the a) massive and b) massless case. The $+$ in b) means sum as vector spaces, but not as representations.

Consider again vectorfields as one-forms. Instead of studying (3.12) at roots of unity, we can study the tensor product of compact representations, using a shift. In particular for $\lambda = -(E_0 - M)\beta_3$, consider $V(\lambda) \otimes V_5$ at roots of unity. For $E_0 > 2$, one can easily see that

$$V(\lambda) \otimes V_5 = V(\lambda + \beta_2) \oplus V(\lambda - \beta_3) \oplus V(\lambda + \beta_3), \quad (3.45)$$

which is the same as for generic q . This follows by analyticity from the generic case (this is why we consider the shifted representations), since the representations on the rhs remain irreducible at $q = e^{i\pi/M}$ according to section 3.2.4 (notice e.g. that the Drinfeld-casimir is different on these representations).

In the (shifted) massless case $E_0 = 2$ however, the generic highest weight representation $V(\lambda + \beta_2)$ has an invariant subspace as in Theorem 3.2.7. In fact, $V(\lambda + \beta_2)$ and $V(\lambda - \beta_3)$ (the "radial" part according to section 3.1.2) are combined in one reducible, but indecomposable representation, similar as the representations I_z^p encountered in section 3.2.1; see figure 3.8 for the noncompact case. This kind of phenomenon at roots of unity is well-known, cp. the general discussion in section 1.2.3. It will be analyzed in general in the next chapter, but we can understand it more directly here.

As explained in 1.2.3, \mathcal{R} is well-defined for $q = e^{i\pi/M}$ if acting on irreps with

$(X_i^\pm)^{M(i)} = 0$ such as the compact $V(\lambda)$ or $V_{(E_0, s)}$. Therefore the invariant sesquilinear form (1.89) on $V(\lambda) \otimes V_5$,

$$(v_1 \otimes w_1, v_2 \otimes w_2)_{\mathcal{R}} = (v_1 \otimes w_1, \mathcal{R}v_2 \otimes w_2)_{\otimes} \quad (3.46)$$

is well-defined and nondegenerate for $q = e^{i\pi/M}$ (but not positive definite in general).

Now it is easy to see that for $q = e^{i\pi/M}$ and $\lambda = (M - 2)\beta_3$, $V(\lambda + \beta_2)$ and $V(\lambda - \beta_3)$ on the rhs of (3.45) are combined into one indecomposable representation: Let $w_{\lambda+\beta_2}$ resp. $w_{\lambda-\beta_3}$ be their h.w. vectors for generic q . We know from Theorem 3.2.7 that for $q = e^{i\pi/M}$, $V(\lambda + \beta_2)$ contains a descendant h.w. vector $\varphi_{\lambda-\beta_3}$ at weight $\lambda - \beta_3$, which means that $\varphi_{\lambda-\beta_3}$ is orthogonal to *any* descendant of $w_{\lambda+\beta_2}$, for any invariant sesquilinear form (while it could happen that this h.w. vector is zero in a given representation, it can be seen easily that this is not the case here). But since our $(\ , \)_{\mathcal{R}}$ is nondegenerate, there must be some vector $\chi \notin V(\lambda + \beta_2)$ which is not orthogonal to $\varphi_{\lambda-\beta_3}$. Since $w_{\lambda+\beta_2}$ is analytic at $q = e^{i\pi/M}$ and $w_{\lambda-\beta_3}$ is orthogonal to $V(\lambda + \beta_2)$ for $|q| = 1$, this is only possible if $w_{\lambda-\beta_3} \rightarrow \varphi_{\lambda-\beta_3} \in \mathcal{U}^- w_{\lambda+\beta_2}$ as $q \rightarrow e^{i\pi/M}$, i.e. the generically independent h.w. modules $V(\lambda - \beta_3)$ and $V(\lambda + \beta_2)$ become dependent, so that one has to include different states such as χ and its descendants to span the tensor product at roots of unity. Furthermore, notice that χ cannot be a h.w. vector because $(\varphi_{\lambda-\beta_3}, \chi)_{\mathcal{R}} \neq 0$, so $X^+ \chi \in V(\lambda + \beta_2)$, and the structure is really indecomposable. This is the reason for the appearance of indecomposable representations.

BRST operator. At first, this may look complicated. However, the main point is that this is actually very nice from the BRST point of view: In fact, the BRST operator Q which was defined "by hand" in the generic case is now an element of the center of \mathcal{U} , and maps χ into $\varphi_{\lambda-\beta_3}$. This is exactly what one would like in QFT. It is not so surprising, since $X^+ \chi \in V(\lambda + \beta_2)$, so some $Q \approx \sum X^- X^+$ should do the job. We will show that

$$Q \equiv (v^{2M} - v^{-2M}) \quad (3.47)$$

is indeed a BRST operator. One could take higher powers of v as well.

To see this, consider first the characteristic equation (1.87) of v in the representation $V(\lambda) \otimes V_5$ for generic q :

$$(v - q^{-c_{\lambda+\beta_2}})(v - q^{-c_{\lambda-\beta_3}})(v - q^{-c_{\lambda+\beta_3}}) = 0. \quad (3.48)$$

For compact representations, $c_\lambda = (\lambda, \lambda + \rho) \in \frac{1}{2}\mathbb{Z}$. As $q \rightarrow e^{i\pi/M}$, the first two factors above coincide, and $(v^{4M} - 1)^2$ contains all the factors in (3.48) separately (and more). So by continuity it follows that $(v^{4M} - 1)^2 = 0$, which implies

$$Q^2 = 0 \quad \text{for} \quad q = e^{i\pi/M} \quad (3.49)$$

on $V(\lambda) \otimes V_5$, since v is invertible.

It remains to show that Q maps some χ into $\varphi_{\lambda-\beta_3}$ for $\lambda = (M-2)\beta_3$. We have seen that as $h \equiv (q - e^{i\pi/M}) \rightarrow 0$, $w_{\lambda-\beta_3}$ becomes $\varphi_{\lambda-\beta_3} \in \mathcal{U}^- w_{\lambda+\beta_2}$, so there is a (fixed) $u^- \in \mathcal{U}$ such that $u^- w_{\lambda+\beta_2} = w_{\lambda-\beta_3} + h\tilde{\chi}(h)$, where $\tilde{\chi}(h) \in V(\lambda) \otimes V_5$ is analytic in h . Applying Q to that equation for generic q , we get

$$Q\left(\frac{u^- w_{\lambda+\beta_2} - w_{\lambda-\beta_3}}{h}\right) = Q\tilde{\chi}(h), \quad (3.50)$$

wich using (1.57) becomes

$$\begin{aligned} -4M(c_{\lambda+\beta_2} u^- w_{\lambda+\beta_2} - c_{\lambda-\beta_3} w_{\lambda-\beta_3}) + o(h) &= -4M(c_{\lambda+\beta_2} - c_{\lambda-\beta_3}) w_{\lambda-\beta_3} + o(h) \\ &= Q(\tilde{\chi}(h)). \end{aligned} \quad (3.51)$$

Now $w_{\lambda-\beta_3} = \varphi_{\lambda-\beta_3} + o(h)$, and therefore for $h = 0$,

$$\varphi_{\lambda-\beta_3} = Q\chi \quad (3.52)$$

where $\chi = c^{-1}\tilde{\chi}(0)$ and $c = -4M(c_{\lambda+\beta_2} - c_{\lambda-\beta_3}) \neq 0$ using Lemma 1.2.1. Thus 2) and 3) of the properties of a BRST operator in section 3.1.2 hold for our Q , and 1) would certainly be satisfied in any covariant theory based on $SO_q(2,3)$. States on which Q does not vanish will be called "ghosts". It is obvious that all this generalizes to higher spins, and works at roots of unity only.

Chapter 4

Operator Algebra and BRST Structure at Roots of Unity

In this chapter, we study the tensor product of irreducible representations for any quantum group at roots of unity using a BRST operator Q , generalizing results of the last section. Q allows us to define in a very simple way a new algebra of representations of \mathcal{U} similar as in QFT, which has a "ghost-free" subalgebra with involution. For the AdS group, this generalizes the physical many-particle Hilbert space introduced above, and can be used to define correlators. Finally we give a conjecture on complete reducibility, generalizing the standard truncated tensor product used in CFT [40]. The problem of a symmetrization postulate is also briefly discussed.

4.1 Indecomposable Representations and BRST Operator

Let V_i be compact irreps for generic q , i.e. with dominant integral h.w. μ_i , which remain irreducible at the root of unity

$$q_0 = e^{2\pi n/m}. \quad (4.1)$$

By a simple shift, this also covers e.g. the unitary representations of $SO_q(2,3)$, but excludes the massless representations for the AdS case (they can be built by tensor products). Then for generic q ,

$$V \equiv V_1 \otimes \dots \otimes V_l = \oplus V(\lambda_l) \quad (4.2)$$

with irreps $V(\lambda_l)$ because it is completely reducible. The space V is the same for any q , and the representation of \mathcal{U} depends analytically on

$$h \equiv q - q_0. \quad (4.3)$$

The projectors P_{λ_l} (1.88) may have a pole at $h = 0$, but no worse singularity. If q is a phase, consider the involution $\overline{x}^c = \theta(x^*)$ as in (1.74).

Now for $q_0 = e^{2\pi i n/m}$ with $M = m$ if m is odd and $M = m/2$ if m is even, the BRST operator Q is defined as in section 3.2.6,

$$Q \equiv (v^{dM} - v^{-dM}) \quad (4.4)$$

where d is an integer depending on the group, such that $q_0^{2dMc_\lambda} = 1$ for any integral weight λ ; notice that c_λ is a rational number.

Of course Q can be considered for any q . For $|q| = 1$, it satisfies

$$\overline{Q}^c = -Q, \quad (4.5)$$

since $\overline{v}^c = v^{-1}$, see (1.77).

If V is completely reducible at q_0 , then Q vanishes at $h = 0$ by construction, using (1.57). In general, it follows that all the eigenvalues of Q are zero at $h = 0$. This implies that $Q^N = 0$ for large enough N (depending on the representation), however Q will not be zero on the indecomposable representations. It is not clear whether $Q^2 = 0$ in general, but this is not essential.

As in the previous section, our essential tools will be the sesquilinear forms $(a, b)_\otimes$ and $(a, b)_\mathcal{R}$ on V defined in section 1.2.1, for $a = a_1 \otimes \dots \otimes a_l \in V_1 \otimes \dots \otimes V_l$ and similarly $b \in V$. Furthermore, define

$$(a, b)_k \equiv (a, Q^k b)_\mathcal{R} \quad (4.6)$$

for $k \in \mathbb{N}$, which is invariant but degenerate. This will play an important role in analyzing the structure of V . If q is a phase, then (4.5) implies

$$Q^\dagger = -Q, \quad (4.7)$$

where † is the operator adjoint w.r.t. any of these invariant inner products.

Now define vectorspaces $G_k \subset V$ as follows:

$$G_k \equiv \{a_k \in V; \quad Q^{k+1}a_k = 0 \text{ at } h = 0\} \quad (4.8)$$

(again, V does not depend on q , only the representation does). At $h = 0$, $Q^N = 0$ for sufficiently large N because of (1.87). Since Q is a Casimir, the G_k are invariant under the action of \mathcal{U} at $h = 0$, and $G_0 \subset G_1 \dots \subset G_k \subset \dots \subset G_N = V$ for some N . This is a "filtration"; it is well known that the tensor product at roots of unity has the structure of a filtration [6]. We can now define the quotients

$$\mathcal{G}_k = G_k / G_{k-1} \quad (4.9)$$

which are representations again (for $h = 0$), and as vectorspace,

$$V \cong \oplus_k \mathcal{G}_k \quad (4.10)$$

(but not as representation). The \mathcal{G}_k with $k > 0$ will be called "ghosts". Roughly speaking, \mathcal{G}_k essentially contains the a_k with $Q^k a_k \neq 0$, but $Q^{k+1} a_k = 0$. From the definition, Q maps \mathcal{G}_k into \mathcal{G}_{k-1} at $h = 0$, and it is injective (i.e. it never vanishes). In particular, $Q^k : \mathcal{G}_k \rightarrow G_0$ is injective as well, and $Q(G_0) = 0$. Therefore at $h = 0$, $(a, b)_k = 0$ if either a or $b \in G_{k-1}$, which means that $(\ , \)_k$ is a well-defined sesquilinear form on \mathcal{G}_k or more generally on $V/G_{k-1} \cong \oplus_{l \geq k} \mathcal{G}_l$.

From now we will work with these spaces for $q = q_0$. Consider v acting on V in its Jordan normal form, $v = D + N$ where D is a diagonal matrix and N nilpotent, and $DN = ND$. Since v is invertible, its (generalized) eigenvalues are nonzero, and D is invertible. Then $v^n = D^n + nD^{n-1}N + N^2(\dots)$ is the Jordan normal form of v^n , so v is diagonalizable if and only if v^n is. In particular for $a \in G_0$, $Qa = (v^{dM} - v^{-dM})a = 0$, therefore $(v^{2dM} - 1)a = 0$ since v is invertible. This means that a is an eigenvector of v^{2dM} , and therefore a is contained in the sum of the *proper* eigenspaces of v , as we have just seen. Conversely of course, any eigenvector of v will be annihilated by Q , since the eigenvalues and the characteristic equation of v are analytic. Therefore we see again that Q is nilpotent on V . In summary,

Lemma 4.1.1 *For $h = 0$,*

$$Q : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1} \quad \text{is injective,} \quad (4.11)$$

and G_0 is the sum of the proper eigenspaces of v .

Now one can define

$$\mathcal{H}_k = \mathcal{G}_k / (Q\mathcal{G}_{k+1}). \quad (4.12)$$

generalizing (3.8); we do not require that $Q^2 = 0$. Since $Q\mathcal{G}_{k+1}$ is invariant, \mathcal{H}_k is a representation of \mathcal{U} as well. An element $h_k \in \mathcal{H}_k$ can be represented by an element of

\mathcal{G}_k (not uniquely of course), which will still be denoted as h_k . Using our sesquilinear forms, one can choose them in a particular way:

Theorem 4.1.2 *Consider $V = V_1 \otimes \dots \otimes V_l$ as above for $h = 0$, with $V \cong \bigoplus_{k=0}^{k_0} \mathcal{G}_k$ as vector space (but not as representation). Then one can choose $H_l \subset \mathcal{G}_l$ such that $(\cdot, \cdot)_k$ is nondegenerate on H_k , and*

$$(H_k, H_{k+l})_k = 0 \quad \text{for } l \geq 1. \quad (4.13)$$

Furthermore,

$$\mathcal{G}_k = H_k \oplus QH_{k+1} \oplus \dots \oplus Q^{k_0-k} H_{k_0} \quad (4.14)$$

as vector space. Therefore $\mathcal{H}_k \cong H_k$, and since $(H_k, Q(\dots))_k = 0$, V/G_{k-1} is the direct sum of H_k and its orthogonal complement w.r.t $(\cdot, \cdot)_k$, as vectorspace.

Notice that this is not trivial, since the $(\cdot, \cdot)_k$ are not positive definite in general. Furthermore, if $Q \neq 0$, then the representation is indecomposable.

Proof Let k_0 be the maximal k such that $G_k \neq 0$. Then for any $h_{k_0} \in H_{k_0} \equiv \mathcal{G}_{k_0}$, $Q^{k_0} h_k \neq 0$, and since $(\cdot, \cdot)_0 = (\cdot, \cdot)_{\mathcal{R}}$ is nondegenerate, there exists some a such that $(a, Q^{k_0} h_{k_0})_0 = (a, h_{k_0})_{k_0} \neq 0$. Furthermore from the definition of $(\cdot, \cdot)_k$ it follows that $(\mathcal{H}_k, G_l)_k = 0$ if $l < k$. Therefore $(\cdot, \cdot)_{k_0}$ is nondegenerate on H_{k_0} . This implies as usual that any linear form on H_{k_0} can be written as $(h_{k_0}, \cdot)_{k_0}$ with a suitable $h_{k_0} \in H_{k_0}$, i.e. there is a isomorphism from the dual space $H_{k_0}^*$ to H_{k_0} .

In particular, any $a_0 \in \mathcal{G}_0$ defines a linear form on H_{k_0} by $h_{k_0} \rightarrow (a_0, h_{k_0})_0$. Therefore there is a unique element $i(a_0) \in H_{k_0}$ which satisfies

$$(a_0 - Q^{k_0-k} i(a_0), H_{k_0})_k = 0. \quad (4.15)$$

Now define $W_0^{k_0} \subset G_0$ by

$$W_0^{k_0} \equiv (id - Q^{k_0} i) G_0; \quad (4.16)$$

then by definition $(W_0^{k_0}, H_{k_0})_0 = 0$, and since $a_0 = (a_0 - Q^{k_0} i(a_0)) + Q^{k_0} i(a_0) \in W_0^{k_0} \oplus Q^{k_0} H_{k_0}$ for any $a_0 \in G_0$, we have

$$G_0 = W_0^{k_0} \oplus Q^{k_0} H_{k_0} \quad \text{with } (W_0^{k_0}, H_{k_0})_0 = 0, \quad (4.17)$$

and $W_0^{k_0}$ is determined uniquely by this requirement.

Now we can define $W_k^{k_0} \subset \mathcal{G}_k$ for any k to be the space which is mapped into $W_0^{k_0}$ by Q^k ,

$$W_k^{k_0} = (Q^k)^{-1} W_0^{k_0} \quad (4.18)$$

(as set). For any $a_k \in \mathcal{G}_k$, decompose $Q^k a_k = w_0^{k_0} + Q^{k_0} h_{k_0}$ accordingly. Then $w_0^{k_0} = Q^k(a_k - Q^{k_0-k} h_{k_0})$, and we have $a_k = w_k^{k_0} + Q^{k_0-k} h_{k_0}$ for $w_k^{k_0} \equiv a_k - Q^{k_0-k} h_{k_0}$. This means that

$$\mathcal{G}_k = W_k^{k_0} \oplus Q^{k_0-k} H_{k_0}, \quad \text{and} \quad (W_k^{k_0}, H_{k_0})_k = 0 \quad (4.19)$$

using the definition of $(\cdot, \cdot)_k$. This decomposition is unique since $Q^k : \mathcal{G}_k \rightarrow G_0$ is injective.

Now for \mathcal{G}_{k_0-1} , define

$$H_{k_0-1} \equiv W_{k_0-1}^{k_0}. \quad (4.20)$$

Then obviously $\mathcal{H}_{k_0-1} \cong H_{k_0-1}$, and $(H_{k_0-1}, H_{k_0})_{k_0-1} = 0$ using (4.19). Thus we have shown (4.14) and (4.13) for $k = k_0 - 1$. Of course, $(H_{k_0-1}, Q(\dots))_{k_0-1} = 0$.

Now we can repeat this construction: since $(H_{k_0-1}, H_{k_0})_{k_0-1} = 0$ and $Q^{k_0-1} h_{k_0-1} \neq 0$ for any $h_{k_0-1} \in H_{k_0-1}$, it follows that $(\cdot, \cdot)_{k_0-1}$ is nondegenerate on H_{k_0-1} . Again, this means that for every $a_0 \in W_0^{k_0}$ there exists some $h_{k_0-1} = i(a_0) \in H_{k_0-1}$ such that $(a_0 - Q^{k_0-1} i(a_0), H_{k_0-1})_0 = 0$. Define $W_0^{k_0-1} \equiv (id - Q^{k_0-1} i) W_0^{k_0}$; then

$$W_0^{k_0} = W_0^{k_0-1} \oplus Q^{k_0-1} H_{k_0-1} \quad (4.21)$$

and $(W_0^{k_0-1}, H_{k_0-1})_0 = 0$. Therefore

$$G_0 = W_0^{k_0-1} \oplus Q^{k_0-1} H_{k_0-1} \oplus Q^{k_0} H_{k_0}, \quad (4.22)$$

and we already know $(W_0^{k_0-1}, H_{k_0})_0 = 0$. This implies that for $W_k^{k_0-1} \equiv (Q^k)^{-1} W_0^{k_0-1}$, $W_k^{k_0} = W_k^{k_0-1} \oplus Q^{k_0-1-k} H_{k_0-1}$ and $(W_k^{k_0-1}, H_{k_0-1})_k = 0$. Finally with

$$H_{k_0-2} \equiv W_{k_0-2}^{k_0-1}, \quad (4.23)$$

(4.14) and (4.13) follows. Repeating this argument, we arrive at the decomposition as stated. \square

4.1.1 A Conjecture on BRST and Complete Reducibility

There are many indications that the following extension of Theorem 4.1.2 holds, in the same context:

Conjecture 4.1.3 *1) \mathcal{H}_0 is completely reducible, i.e. it is the direct sum of irreducible representations.*

2) \mathcal{H}_0 defines an associative, completely reducible modified tensor product of irreducible representations.

Furthermore, it appears that G_0 is the sum of the analytic images of the projectors P_λ at $h = 0$, see Lemma 1.2.3, and $G_0 \cap Q(\dots)$ is the subspace where they are linearly dependent. This could be used to show 2) from 1).

In the context of the AdS group, 2) would generalize the definition of the tensor product $\tilde{\otimes}$ in sections 3.2.2 and 3.2.5, and include the "high-energy" bands as well as irreps which are unitary w.r.t. the "compact" reality structure. This would also generalize the standard truncated tensor product used in the context of CFT, and provide a common framework for $\tilde{\otimes}$ and $\hat{\otimes}$.

4.2 An Inner Product on G_0

In this section, we will define a hermitian inner product on G_0 . To do this, we first show how to find an element in \mathcal{U} which implements \sqrt{v} (or many other functions of v) analytically on G_0 for $q = q_0$.

As already noted, v becomes degenerate on different representations at $h = 0$, because q^{-c_λ} does. For a weight λ , define the λ -group g_λ of V to be the set of highest weights in the generic decomposition (4.2) of V for which v is degenerate,

$$g_\lambda \equiv \{\lambda_l; \quad q_0^{c_{\lambda_l}} = q_0^{c_\lambda}\}. \quad (4.24)$$

Then define

$$P_{g_\lambda} \equiv \frac{\prod_{g_{\lambda'} \neq g_\lambda} (v - q^{-c_{\lambda'}})}{\prod_{g_{\lambda'} \neq g_\lambda} (q^{-c_\lambda} - q^{-c_{\lambda'}})}, \quad (4.25)$$

where the products go over all possible λ -groups once. As opposed to P_λ , this is analytic at q_0 . It is not a projector in V , but it *is* a projector for $h = 0$ in G_0 , since then v is diagonalizable on G_0 as pointed out in Lemma 4.1.1; it is simply the projection on the eigenspaces of v corresponding to different λ -groups.

Now define

$$\sqrt{v} \equiv \sum_{g_\lambda} P_{g_\lambda} q^{-\frac{1}{2}c_\lambda}, \quad (4.26)$$

which is an element of \mathcal{U} and analytic at q_0 , where λ is some element of g_λ . It depends on the choice of $\lambda \in g_\lambda$, which simply corresponds to choosing a different branch; pick any of them. It is easy to see that on G_0 ,

$$(\sqrt{v})^2 = v \quad \text{for } h = 0, \quad (4.27)$$

and its inverse on G_0 is given by

$$\sqrt{v^{-1}} \equiv \sum_{g_\lambda} P_{g_\lambda} q^{\frac{1}{2}c_\lambda}. \quad (4.28)$$

This follows because the P_{g_λ} are projectors on G_0 as shown above.

Using this, we can finally show the following:

Proposition 4.2.1 *Define*

$$(a, b)_H \equiv c_V(a, \sqrt{v} \cdot b)_\mathcal{R}, \quad (4.29)$$

where $c_V = q^{\frac{1}{2}(c_1 + \dots + c_l)}$ and c_i is the quadratic Casimir on the V_i . Then for $a, b \in G_0$, $(a, b)_H$ is hermitian at $h = 0$, i.e. it is an inner product on G_0 for $q = q_0$.

Proof To see this, notice first that

$$\Delta_{(l)} v = (\mathcal{R}_{l\dots 21} \mathcal{R}_{12\dots l})^{-1} (v \otimes \dots \otimes v). \quad (4.30)$$

This follows inductively by applying $(\Delta_{(l-1)} \otimes \text{id})$ to (1.53):

$$\begin{aligned} \Delta_{(l)}(v) &= ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{21} (\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{12})^{-1} (\Delta_{(l-1)} v \otimes v) \\ &= ((\Delta_{(l-1)} \otimes \text{id}) \mathcal{R}_{21} (\mathcal{R}_{12\dots(l-1)}^{-1} \otimes 1) \mathcal{R}_{12\dots l})^{-1} (\Delta_{(l-1)} v \otimes v) \\ &= ((\mathcal{R}_{12\dots(l-1)}^{-1} \otimes 1) ((\Delta'_{(l-1)} \otimes 1) \mathcal{R}_{21}) \mathcal{R}_{12\dots l})^{-1} (\Delta_{(l-1)} v \otimes v) \\ &= ((\mathcal{R}_{12\dots(l-1)}^{-1} \otimes 1) (\mathcal{R}_{(l-1)\dots 21}^{-1} \otimes 1) \mathcal{R}_{l\dots 21} \mathcal{R}_{12\dots l})^{-1} (\Delta_{(l-1)} v \otimes v) \\ &= (\mathcal{R}_{l\dots 21} \mathcal{R}_{12\dots l})^{-1} (\mathcal{R}_{(l-1)\dots 21} \mathcal{R}_{12\dots(l-1)} \otimes 1) (\Delta_{(l-1)} v \otimes v) \\ &= (\mathcal{R}_{l\dots 21} \mathcal{R}_{12\dots l})^{-1} (v \otimes \dots \otimes v) \end{aligned} \quad (4.31)$$

using (1.58), (1.59) and Lemma 1.1.1. Furthermore $\overline{\sqrt{v}}^c = \sqrt{v^{-1}}$. Then

$$\begin{aligned} c_V(a, b)_H^* &= (a, \mathcal{R}_{12\dots l} \Delta_{(l)}(\sqrt{v})b)_\otimes^* \\ &= (\mathcal{R}_{12\dots l} \Delta_{(l)}(\sqrt{v})b, a)_\otimes \\ &= (\Delta'_{(l)}(\sqrt{v}) \mathcal{R}_{12\dots l} b, a)_\otimes \\ &= (b, \mathcal{R}_{l\dots 21}^{-1} \Delta_{(l)}(\sqrt{v^{-1}})a)_\otimes \\ &= (b, \mathcal{R}_{12\dots l} \mathcal{R}_{12\dots l}^{-1} \mathcal{R}_{l\dots 21}^{-1} v^{-1} \cdot \sqrt{v} \cdot a)_\otimes \\ &= (b, (v^{-1} \otimes \dots \otimes v^{-1}) \sqrt{v} \cdot a)_\mathcal{R} \\ &= c_V(b, a)_H \end{aligned} \quad (4.32)$$

for $h = 0$, using (1.61). \square

The above definitions may seem a bit complicated at first. We will show below that all this can be formulated in a very simple way, using an extension of \mathcal{U} by a universal element which implements a Weyl reflection. But before that, we show how the above inner product defines a *Hilbert space* of physical many-body representations of the Anti-de Sitter group, with the correct classical limit. Then the adjoint of an operator acting on any component of a tensor product is determined by the positive definite inner product, and is guaranteed to have the correct classical limit, since the inner product has (for $q \neq 1$, its adjoint will act on the entire tensor product, see below). This finally settles any doubts whether the reality structure (1.74) is suitable to describe physical many-body states.

4.2.1 Hilbert Space for the Quantum Anti-de Sitter group

It follows from Theorem 4.1.2 that $(a, Q(\dots))_H = 0$ for $a \in G_0$, i.e. the image of Q is null with respect to this inner product. Therefore $(\ , \)_H$ induces an invariant inner product on H_0 as defined in section 4.1. Furthermore, $(\ , \)_H$ is nondegenerate on H_0 . We want to show that it induces in fact a *positive definite* inner product on the physical representations

$$V_{(\mu_1)} \tilde{\otimes} \dots \tilde{\otimes} V_{(\mu_l)} = \bigoplus_{\lambda_k} V_{(\lambda_k)} \quad (4.33)$$

defined in section 3.2.5, where all $V_{(\mu_l)}$ are massive (this is not an essential restriction). The terms on the rhs are (quotients of) analytic, generic representations and therefore contained in (a quotient of) G_0 . Thus the results of the last section apply, and the eigenvalues of $(\ , \)_H$ on G_0 are either positive or negative. It is clear that for low energies, they will be positive as in the classical limit. We claim that they are positive on all the $V_{(\lambda_k)}$ above, and outline a proof:

Consider the tensor product V of the infinite-dimensional (generic) lowest-weight modules corresponding to the massive representations in (4.33). For fixed λ_k , let G_{λ_k} be the sum of the lowest-weight submodules of V with lowest weight λ_k . According to the strong linkage principle, none of them will contain physical lowest-weight vectors, for any q on the arc from 1 to $e^{2i\pi/m}$. This implies that the physical lowest-weight vectors of G_{λ_k} are linearly independent of the other $G_{\lambda'_k}$ for q on the arc from 1 to $q = e^{2i\pi/m}$, and analytic in q .

Now we can define $(\ , \)_H$ as in (4.29) for states with weights λ_k corresponding

to physical G_{λ_k} , where one has to use the value of the classical quadratic Casimir on G_{λ_k} . \mathcal{R} is well-defined for such states, as can be seen from (1.47) and the discussion in section 1.2.3. Therefore $(\ , \)_H$ is hermitian and analytic for q on the arc from 1 to $e^{2i\pi/m}$. For $q = e^{2i\pi/m}$, it reduces precisely to our inner product on G_0 . Furthermore, all states with such weights are contained in $\oplus G_{\lambda_k}$.

Now $(\ , \)_H$ cannot become null on the physical lowest-weight states of G_{λ_k} , where it is non-degenerate for q on the arc from 1 to $e^{2i\pi/m}$, since the G_{λ_k} remain linearly independent, so \sqrt{v} is analytic and invertible, and the other $G_{\lambda_{k'}}$ are orthogonal. Therefore the eigenvalues of $(\ , \)_H$ on G_{λ_k} are positive, as classically.

4.3 The Universal Weyl Element w

The proper mathematical tool to obtain this inner product and an involution is an element of an extension of \mathcal{U} by generators of the braid group ω_i , introduced in [33] and [34]. The w_i act on representations of \mathcal{U} and implement the braid group action (1.46) on \mathcal{U} via $T_i(x) = w_i x w_i^{-1}$ for $x \in \mathcal{U}$. All we need is the generator corresponding to the longest element of the Weyl group, w . Acting on a highest weight irrep, w maps the highest weight vector into the lowest weight vector of the contragredient representation. It has the following important properties [33, 34]:

$$\Delta(w) = \mathcal{R}^{-1} w \otimes w = w \otimes w \mathcal{R}_{21}^{-1} \quad (4.34)$$

$$w^2 = v\varepsilon \quad (4.35)$$

where ε is a Casimir with

$$\Delta(\varepsilon) = \varepsilon \otimes \varepsilon \quad (4.36)$$

(for $SL_q(2)$ and $SO_q(2,3)$, ε is +1 for integer "spin" and -1 for half-integer "spin", see Appendix B). (4.34) justifies the name "universal Weyl element". One can also find the antipode and counit of w . Furthermore, for the "real" quantum groups (=those having only self-dual representations, i.e. all except A_n , D_n and E_6), such as $SO_q(5, C)$ and its real forms, the following holds [49]:

$$wxw^{-1} = \theta Sx = S^{-1}\theta x. \quad (4.37)$$

We will only consider this "real" case. For "complex" groups, this gets corrected by an automorphism of the Dynkin diagram. Actually, (4.37) and (4.35) have only been proved explicitly for the $SL_q(2)$ case in the literature, therefore we will supply proofs in Appendix B.

Similarly, we define

$$\tilde{w} \equiv (-1)^E q^{2\bar{\rho}} w \quad (4.38)$$

with the same properties except

$$\tilde{w}x\tilde{w}^{-1} = \tilde{\theta}S^{-1}x = S\tilde{\theta}x. \quad (4.39)$$

Here $\tilde{\theta}x \equiv (-1)^E \theta(x)(-1)^E$ implements the reality structure on \mathcal{U} according to section 1.1.5: in the case of $SO_q(2, 3)$, E is the energy operator, while the compact case corresponds to $E = 0$.

Denote the (left) action of $x \in \mathcal{U}$ on a representation V by

$$x \triangleright v_i = v_j \pi_i^j(x), \quad (4.40)$$

for a basis v_i of V . We are mainly interested in (tensor products of) unitary representations. The following will be very useful:

Lemma 4.3.1 *If an irrep $V(\lambda)$ is analytic and unitary at a phase q (for the compact involution (1.73), say), then it has a basis which is orthogonal and normalized w.r.t. both its symmetric bilinear form and its invariant inner product, i.e.*

$$\pi_j^i(\theta(x)) = \pi_i^j(x) \quad \text{and} \quad (4.41)$$

$$\pi_j^i(\bar{x}^c) = \pi_i^j(x)^* \equiv (\pi_i^j(x))^*, \quad (4.42)$$

in that basis.

Proof First, one can check that this holds for the fundamental representations V_f (i.e. the spinor representation for $SO_q(5)$) [49]. We will show the general statement inductively by taking tensor products with the fundamental representation. Suppose it holds for π_j^i on $V(\mu)$, and consider $V(\mu) \otimes V_f = \oplus V(\lambda_i)$. All the multiplicities are known to be one in this case (this can be seen e.g. using the Racah–Speiser algorithm). Then the Clebsch–Gordan coefficients $K_m^{ij}(q)$ defined by $v_m^{(\lambda_i)} = K_m^{ij} v_i \otimes v_j \in V(\lambda_i)$ satisfy [49]

$$K_m^{ij}(q) R_{ij}^{ls} = (-1)^{\nu_i} q^{\frac{1}{2}(c_{\lambda_i} - c_\mu - c_f)} \widetilde{K}_m^{sl}(q), \quad (4.43)$$

$$K_m^{ij}(q^{-1}) = (-1)^{\nu_i} \widetilde{K}_m^{ji}(q). \quad (4.44)$$

Here $\widetilde{K}_m^{sl}(q)$ is the Clebsch in the reversed tensor product, and $(-1)^{\nu_i}$ is the same as classically (and just a convention unless the factors are identical). The first can be seen similarly as in section 1.2.2. Furthermore, $K_m^{ij}(q)$ is real for $q \in \mathbb{R}$.

Using this, we can write an invariant inner product of 2 vectors in $V(\lambda_i)$ as in (4.29),

$$\begin{aligned}
(v_m^{(\lambda_i)}, v_n^{(\lambda_i)})_{\mathcal{H}} &= q^{-\frac{1}{2}(c_{\lambda_i} - c_{\mu} - c_f)} K_m^{ij}(q^{-1}) K_n^{st}(q) R_{st}^{kl}(v_i \otimes v_j, v_k \otimes v_l)_{\otimes} \\
&= \widetilde{K}_m^{ji}(q) \widetilde{K}_n^{lk}(q) \delta_k^i \delta_l^j \\
&= \delta_n^m,
\end{aligned} \tag{4.45}$$

choosing an orthogonal basis of $V(\lambda_i)$, which is unitarizable by assumption. But then the invariant bilinear form of these vectors is

$$\begin{aligned}
(v_m^{(\lambda_i)}, v_n^{(\lambda_i)})_{\otimes}^{(bi)} &= K_m^{ij}(q) K_n^{kl}(q) (v_i, v_k)^{(bi)} (v_j, v_l)^{(bi)} \\
&= \delta_n^m,
\end{aligned} \tag{4.46}$$

using $(v_i, v_k)^{(bi)} = \delta_k^i$ on the components, by the induction assumption. This means that the $v_m^{(\lambda_i)}$ are orthogonal not only w.r.t. the sesquilinear inner product, but also the above bilinear form. \square

In the case of $SU_q(2)$, this can easily be checked explicitly.

We will always use this basis from now on. By inserting $(-1)^E$, one gets the same statement for the shifted noncompact representations, with θ replaced by $\tilde{\theta}$. Moreover, the result also holds for the irreducible quotients of analytic representations at roots of unity of a given unitarity type, such as the physical many-particle representations defined in section 3.2.2 and 3.2.5.

Now we can get a better understanding of \tilde{w} . First, it intertwines a unitary representation with its contragredient (=dual) representation, defined by $x \tilde{\triangleright} a_i = a_j \pi_j^i(S^{-1}x)$ (remember that we only consider "real" groups):

$$\begin{aligned}
x \triangleright (\tilde{w} \triangleright a_i) &= (x \tilde{w}) \triangleright a_i = \tilde{w} \triangleright (\tilde{\theta} S^{-1}x) \triangleright a_i \\
&= \tilde{w} \triangleright (a_j \pi_j^i(\tilde{\theta} S^{-1}x)) = \tilde{w} \triangleright (a_j \pi_j^i(S^{-1}x))
\end{aligned} \tag{4.47}$$

$$= \tilde{w} \triangleright (x \tilde{\triangleright} a_i). \tag{4.48}$$

Other important properties are as follows. Define

$$g_{ij} \equiv \pi_j^i(\tilde{w}) q^{\frac{1}{2}c_{\lambda}} = (-1)^f g^{ij} \tag{4.49}$$

where c_{λ} is the classical quadratic Casimir of the representation, and $(-1)^f$ is the value of ε . The last equality follows from (4.35), where $g_{ij} g^{jl} = \delta_i^l$. This is nothing but the invariant tensor (again, for "real" groups):

Proposition 4.3.2

$$\pi_k^i(x_1)\pi_l^j(x_2)g^{kl} = \epsilon(x)g^{ij} \quad (4.50)$$

$$g_{ij}\pi_k^i(x_1)\pi_l^j(x_2) = \epsilon(x)g_{kl} \quad (4.51)$$

$$g_{ij} = q^{\frac{1}{2}c\lambda}\pi_i^j((-1)^E w) \quad (4.52)$$

$$g_{ls}\hat{R}_{ut}^{sk}g^{tv} = (\hat{R}^{-1})_{lu}^{kv}. \quad (4.53)$$

(4.53) and some more similar relations are contained in [49].

Proof (4.50) follows from

$$\begin{aligned} \pi_k^i(x_1)\pi_l^j(x_2)\pi_l^k(\tilde{w}) &= \pi_l^j(x_2)\pi_l^i(x_1\tilde{w}) \\ &= \pi_l^j(x_2)\pi_n^i(\tilde{w})\pi_l^n(\tilde{\theta}(S^{-1}x_1)) \\ &= \pi_l^j(x_2)\pi_n^i(\tilde{w})\pi_n^l(S^{-1}x_1) \\ &= \pi_n^j(x_2S^{-1}x_1)\pi_n^i(\tilde{w}) \\ &= \epsilon(x)\delta_n^j\pi_n^i(\tilde{w}), \end{aligned} \quad (4.54)$$

and similarly (4.50). (4.52) follows from the uniqueness of the invariant tensor (cp. Lemma 2.2.2), noting that

$$\begin{aligned} \pi_k^i(x_1)\pi_l^j(x_2)\pi_k^l((-1)^E w) &= \pi_k^j(x_2q^{-2\bar{\rho}}\tilde{w})\pi_k^i(x_1) \\ &= \pi_k^j(q^{-2\bar{\rho}}\tilde{w}\tilde{\theta}Sx_2)\pi_k^i(x_1) \\ &= \pi_t^j(q^{-2\bar{\rho}}\tilde{w})\pi_k^t(\tilde{\theta}Sx_2)\pi_k^i(x_1) \\ &= \pi_t^j(q^{-2\bar{\rho}}\tilde{w})\pi_t^k(Sx_2)\pi_k^i(x_1) \\ &= \pi_t^j((-1)^E w)\delta_t^i\epsilon(x) \end{aligned} \quad (4.55)$$

is indeed invariant, where we used (1.51). This shows (4.52) up to a constant, which is one because

$$\begin{aligned} \pi_j^i(\tilde{w})\pi_k^j(\tilde{w}) &= \pi_k^i(v\varepsilon) \quad \text{and} \\ \pi_i^j((-1)^E w)\pi_j^k((-1)^E w) &= \pi_i^k(v\varepsilon), \end{aligned} \quad (4.56)$$

which is the same. Finally, (4.53) is obtained by taking representations of

$$(1 \otimes \tilde{w})\mathcal{R}(1 \otimes \tilde{w}^{-1}) = (1 \otimes \tilde{\theta}S^{-1})\mathcal{R} = (1 \otimes \tilde{\theta})\mathcal{R}^{-1}. \quad (4.57)$$

□

Furthermore, from (4.34) one immediately obtains (1.60), (1.61), as well as (4.30). For $|q| = 1$, it is consistent to define

$$\overline{\tilde{w}} = \tilde{w}^{-1} \quad (4.58)$$

and similarly for w . Correspondingly, we have

$$\pi_j^i(\tilde{w})^* = \pi_i^j(\tilde{w}^{-1}). \quad (4.59)$$

This can be checked explicitly using the known formulas for w in terms of the w_i [33] and their action on a representation [25, 39].

4.4 An Algebra of Creation and Annihilation Operators

In this section, we will show how to define an interesting algebra of creation and annihilation operators, with involution. This allows us to work with states of different particle numbers, and to write down correlators as in QFT. It is however not clear at present how identical particles should be defined, so we only consider distinguishable particles.

Denote the states of a physical representation $V_{(\lambda)}$ of $SO_q(2, 3)^1$ by a_i . We can make $V_{(\lambda)}$ into a \mathcal{U} -module algebra $\mathcal{F}^{(a)}$ (see section 2.1.1) by defining either $a_i a_j = 0$, or more somewhat more interestingly by

$$a_i a_j = g_{ij} a^2, \quad a_i a_j a_k = 0 \quad (4.60)$$

where a^2 is a (scalar) variable, and g_{ij} as in (4.49). We will use this algebra only to show how to define an involution, and to write the inner product on G_0 in a elegant way. Notice that in the noncompact case, $g_{ij} \neq 0$ only if one of the "factors" is a positive and one a negative energy representation (an antiparticle wavefunction, i.e. shifted by $-2M\beta_3$ as explained in section 3.2.4). We are using the same letter a_i , because both positive and negative energy representations can be obtained as quotients of one "big" self-dual representation, namely the \mathcal{H}_0 part of a tensor product as in Theorem 4.1.2, which is the direct sum of a positive and a negative energy representation. This is very nice from the QFT point of view, and one should keep this in mind for the following.

¹ This works for other real groups as well.

Given many left \mathcal{U} -module algebras $\mathcal{F}^{(a)}, \mathcal{F}^{(b)}, \dots$ generated by a_i, b_i, \dots , one can define a combined (left) \mathcal{U} -module algebra \mathcal{F} using the *braided tensor product* [42]. As vector space, this is simply $\mathcal{F} = \mathcal{F}^{(a)} \otimes \mathcal{F}^{(b)} \otimes \dots$, with commutation relations

$$a_i b_j = (\mathcal{R}_2 \triangleright b_j)(\mathcal{R}_1 \triangleright a_i) \gamma \quad (4.61)$$

for some $\gamma \in \mathbb{C}$; similarly for more variables. This definition requires an (arbitrary) "ordering" $a > b > \dots$ of the different algebras. It is consistent because of the standard properties of \mathcal{R} .

Finally on the vector space $\mathcal{U} \otimes \mathcal{F}$, one can define a *cross product* algebra $\mathcal{U} \ltimes \mathcal{F}$ via

$$x a_i = (x_1 \triangleright a_i) x_2, \quad \text{or equivalently} \quad x \triangleright a_i = x_1 a_i S x_2. \quad (4.62)$$

This is an algebra, because \mathcal{U} is a Hopf algebra and \mathcal{F} is a (left) \mathcal{U} -module algebra.

To make the connection with QFT more obvious, we define a ("Fock") vacuum \rangle which reduces $\mathcal{U} \ltimes \mathcal{F}$ to its "vacuum-representation" $\mathcal{U} \ltimes \mathcal{F} \rangle$ by

$$f(a, b, \dots) x \rangle \equiv \epsilon(x) f(a, b, \dots) \rangle. \quad (4.63)$$

This is a tensor product of representation of \mathcal{U} , and

$$x f(a, b, \dots) \rangle = x_1 \triangleright f(a, b, \dots) x_2 \rangle = x \triangleright f(a, b, \dots) \rangle. \quad (4.64)$$

In particular this contains the subspace G_0 where v is diagonalizable, and we can apply the results of sections 4.1 and 4.2.

Define

$$\Omega = \tilde{w} \sqrt{v^{-1}} \quad (4.65)$$

with $\sqrt{v^{-1}}$ as in (4.28). Acting on G_0 , this satisfies

$$\Omega^2 = \varepsilon \quad (4.66)$$

at roots of unity, using (4.35) and $S(v) = v$. Furthermore from (4.58),

$$\overline{\Omega} = \Omega^{-1} \quad (4.67)$$

if acting on G_0 .

Now at roots of unity, define \mathcal{H} to be the algebra $\mathcal{U} \ltimes \mathcal{F}$ with the *additional* relation $Q = 0$, i.e.

$$\mathcal{H} \equiv (\mathcal{U} \ltimes \mathcal{F}) / Q. \quad (4.68)$$

It is closely related to the \mathcal{H}_0 representations of section 4.1, see also Conjecture 4.1.3.

This is very similar to the definition of the physical Hilbert space in QFT using the BRST operator (3.8), formulated in terms of an algebra instead of representations, which is essentially the same. Since Q vanishes on all generic representations, this contains in particular the (semidirect product of \mathcal{U} with the) physical many-particle states in the case of $SO_q(2,3)$, as well as the states of the usual truncated tensor product in the context of CFT.

Using Ω , we can essentially define an involution on \mathcal{H} (resp. for G_0 -type representations of $\mathcal{U} \ltimes \mathcal{F}$) as follows: for $x \in \mathcal{U}$, it is simply the involution \bar{x} corresponding to the reality structure considered. If x is either X_i^\pm or H_i , this can be written as

$$\bar{x} = \Omega S(x) \Omega^{-1}. \quad (4.69)$$

For a generator a_i of \mathcal{F} , we define

$$\bar{a}_i = \Omega a_i \Omega^{-1}, \quad (4.70)$$

and extend this as an antilinear antialgebra-homomorphism on \mathcal{H} (resp. $\mathcal{U} \ltimes \mathcal{F}$). It is shown in Appendix A that this is consistent with the algebra \mathcal{H} provided the a_i are unitary w.r.t. the reality structure on \mathcal{U} , and with the braiding algebra (4.61) if γ is a phase. It is also consistent with (4.60) provided $\bar{a}^2 = (-1)^{f_a} a^2$ where $(-1)^{f_a}$ is the value of ε on a_i ; note that $g_{ij}^* = (-1)^{f_a} g_{ji}$, using (4.59). Again in the noncompact case, Ω can act on a_i only if it contains both positive and negative energy states, such as \mathcal{H}_0 in Theorem 4.1.2. Now

$$\bar{\bar{a}}_i = v \sqrt{v^{-1}}^2 \varepsilon a_i v^{-1} \sqrt{v^{-1}}^{-2} \varepsilon^{-1} \quad (4.71)$$

$$= a_i (-1)^{f_a} \quad (4.72)$$

in \mathcal{H} (resp. on G_0 -type representations), using (4.36) resp. (4.66). This is as good as an involution, and the main result of this section. In fact, one could define instead $\bar{a}_i = \varepsilon \Omega a_i \Omega^{-1}$, which really is an involution; we choose not to do this here. Again, the operator adjoint can be calculated once there is a positive definite inner product. We want to indicate how this could be achieved:

Let \mathcal{F} be as above for braided copies a_i, b_i, \dots of the algebra (4.60). Define an evaluation of \mathcal{H}

$$\langle x f(a, b, \dots) \rangle \equiv \varepsilon(x) \langle f(a, b, \dots) \rangle \quad (4.73)$$

by first collecting the generators of the same algebra using the braiding relations, and defining $\langle a^2 \rangle \equiv (-1)^{f_a} g_a$ with $g_a \in \mathbb{R}$, and $\langle a_j \rangle \equiv 0$; similarly for the other variables.

This is independent of ordering if $\gamma = \pm 1$, because a^2 is then central. It satisfies

$$\langle xf(a, b, \dots) \rangle = \langle f(a, b, \dots)x \rangle = \epsilon(x) \langle f(a, b, \dots) \rangle \quad (4.74)$$

$$\langle f(a, b, \dots) \rangle^* = \overline{\langle f(a, b, \dots) \rangle} (-1)^f. \quad (4.75)$$

on \mathcal{H} where $f = f_a + f_b + \dots$, using $\overline{a^2} = a^2(-1)^{f_a}$.

One can now write states of \mathcal{H}_0 in the form $f\rangle = f^{i_a i_b \dots} a_{i_a} b_{i_b} \dots\rangle$ and $g\rangle = g^{i_a i_b \dots} a_{i_a} b_{i_b} \dots\rangle$ where a_{i_a}, b_{i_b}, \dots are positive-energy (physical) states, and define an inner product as follows:

$$(f, g) \equiv \langle \overline{f} g \rangle. \quad (4.76)$$

This is hermitian, invariant

$$(x \triangleright f, g) = (f, \overline{x} \triangleright g) \quad (4.77)$$

and it is shown in Appendix A that it is in fact the same as the inner product defined in (4.29),

$$(a_i b_j \dots, a_k b_l \dots) \equiv \langle \overline{a_i b_j \dots} a_k b_l \dots \rangle = (a_i \otimes b_j \otimes \dots, a_k \otimes b_l \otimes \dots)_H, \quad (4.78)$$

if the normalization on the rhs is chosen as $(a_i, a_j) = g_a \delta_j^i$ (remember that we work in an orthogonal basis). Invariance can be seen easily:

$$\begin{aligned} (x \triangleright f, g) &= \langle \overline{(x_1 f S x_2)} g \rangle = \langle \overline{S x_2} (\overline{f}) \overline{x_1} g \rangle = \langle \overline{f} \overline{x} g \rangle \\ &= (f, \overline{x} \triangleright g), \end{aligned} \quad (4.79)$$

using (4.74). Hermiticity follows from (4.75). Positivity was discussed in section 4.2.1 for the case of the Anti-de Sitter group.

One could formulate all this without using w explicitly, in the form $\overline{a_i} = (\Omega_1 \triangleright a_i) \Omega_2$ which can be written in terms of the universal \mathcal{R} . Needless to say, this would be much more complicated. However it helps to understand the main point of this definition, namely the \mathcal{R} involved which "corrects" the flipping of the tensor product in the reality structure (1.74). In this form, a somewhat similar-looking conjugation was introduced in [40].

4.4.1 On Quantum Fields and Lagrangians

In this section, we want to show how the above formalism could find application in a QFT. This can only be very vague at present, because an important piece is

still missing – the implementation of a symmetrization postulate, in order to define identical particles. We can nevertheless write down a few generic formulas in an ad–hoc way.

Consider a (large) number of braided copies of the algebra (4.60) with generators $a_i^{\lambda,(n)}$, for $n = 1, 2, \dots, N$ and λ going through all possible highest weights of physical (unitary) representations of a given spin; remember that there exist only finitely many at roots of unity.

Then consider the following object:

$$\Psi(y) \equiv \frac{1}{\sqrt{N}} \sum_{i,n} a_i^{\lambda,(n)} f_{\lambda,(n)}^i(y). \quad (4.80)$$

Here $f_{\lambda,(n)}^i(y)$ is the dual representation of $a_i^{\lambda,(n)}$, realized as functions (or forms, ...) on quantum AdS space, which is a *right* \mathcal{U} –module algebra in the dual picture of chapter 2. This works as in Theorem 4.1.2, where the factors V_i are the 5–dimensional representations y^i . Then the unitary representations are the quotients \mathcal{H}_0 , in the space of functions on quantum AdS space. By this construction, $\Psi(y)$ contains both positive and negative energy representations as discussed earlier, so that Ω can act on it. Thus we can assume that

$$u \triangleright \sum a_i^{\lambda,(n)} f_{\lambda,(n)}^i(y) = \sum a_j^{\lambda,(n)} \pi_i^{j\lambda}(u) f_{\lambda,(n)}^i(y) = \sum a_j^{\lambda,(n)} (f_{\lambda,(n)}^j(y) \triangleleft u). \quad (4.81)$$

for $u \in \mathcal{U}$. $\Psi(y)$ behaves very much like an off–shell quantum field. Similarly, we can consider $\Psi(y_1), \Psi(y_2), \dots$ with (braided) copies y_i of quantum AdS space. Then for example,

$$\langle \Psi(y_1) \Psi(y_2) \rangle = \pm \sum_{\lambda} g_{\lambda} g_{ij} f_{\lambda,(n)}^i(y_1) f_{\lambda,(n)}^j(y_2) + o(\hbar) \quad (4.82)$$

becomes a correlator in the classical limit, depending on the choice of the $g_{\lambda} \equiv g_{a\lambda}$ for the representations involved. In particular, $g_{\lambda} = \frac{i}{\square_{\lambda} - m^2}$ corresponds to a Green’s function, where \square_{λ} is the (quantum) quadratic Casimir (see [18]). More generally,

$$\langle \Psi(y_1) \dots \Psi(y_k) \rangle \quad (4.83)$$

will reproduce the sum of Wick contractions at $q = 1$ for $N \rightarrow \infty$.

Using this, one can write down e.g. an ”interaction Lagrangian”

$$\mathcal{S} = \int_y \Psi(y) \dots \Psi(y), \quad (4.84)$$

where the integral over quantum AdS space is defined in section 2.3.1 (in different notation), and many similar terms. This is invariant under $SO_q(2, 3)$, i.e. $x\mathcal{S} = \mathcal{S}x$

in $\mathcal{U} \ltimes \mathcal{F}$ for $x \in \mathcal{U}$, using the invariance of the integral. The only thing we want to do here is to show that \mathcal{S} is in fact hermitian, with the involution defined in the previous section:

$$\overline{\int_y \Psi(y) \dots \Psi(y)} = \int_y \Psi(y) \dots \Psi(y). \quad (4.85)$$

If one can find an algebra replacing (4.60) such that the inner product (4.76) is positive definite, this implies that $e^{i\mathcal{S}}$ is a unitary operator on H_0 (in the present version, this is not the case because of the additional generator a^2).

To see (4.85), let us simplify the notation first by writing a_i, b_j, c_k for $a_i^{\lambda, (1)}, a_j^{\lambda, (2)}$ and $a_k^{\lambda, (3)}$, respectively, and to be specific consider

$$\tilde{\mathcal{S}} = \int_y \psi_a(y) \psi_b(y) \psi_c(y) \quad (4.86)$$

where $\psi_a(y) \equiv \sum_i a_i f^i(y)$, and similarly $\psi_b(y), \psi_c(y)$. We claim that

$$\overline{\tilde{\mathcal{S}}} = \int_y \psi_c(y) \psi_b(y) \psi_a(y). \quad (4.87)$$

It is clear that this implies (4.85). We consider only scalar fields here for simplicity.

To see (4.87), first observe that

$$\overline{f^i(y)}^b = \pm f^i(y) \quad (4.88)$$

using the (auxiliary) antilinear involution on quantum AdS space defined in section 2.3.1. This follows from $\overline{y^i}^b = -y^i$ and their algebra, (4.43) and (4.44). Together with (2.88), this implies

$$\begin{aligned} \overline{\int_y \psi_a(y) \psi_b(y) \psi_c(y)} &= \sum \Omega c_k b_j a_i \Omega^{-1} \int_y f^k(y) f^j(y) f^i(y) \\ &= \int_y \psi_c(y) \psi_b(y) \psi_a(y) \end{aligned} \quad (4.89)$$

as claimed, since this is a scalar and it therefore commutes with \mathcal{U} and Ω .

One could now go ahead and define "ad hoc" correlators such as $\langle \Psi(y_1) e^{i\mathcal{S}} \Psi(y_2) \rangle$, which in fact can be expressed as sum of contractions as seen above, similar to the classical Wick expansion. Moreover for $q \neq 1$, these contractions can be interpreted naturally as generalized links, with interaction vertices being Clebsch–Gordan coefficients defined by the integral, and crossings given by the \hat{R} -matrices in the particular representations. All these diagrams would be finite, since there exist only finitely many "physical" representations, which are all finite-dimensional. Moreover, these correlators are in fact symmetric in the classical limit, since the $\Psi(y_i)$ commute for

$q = 1$. Nevertheless, this is not really satisfactory, since an explicit symmetrization postulate is missing, as well as a dynamical principle determining "on-shell" states. These two open problems are probably related.

Symmetrization. Let us briefly discuss the problem of symmetrization and identical particles. For generic q , this has been studied in [21]. At roots of unity, the situation can be expected to be somewhat different.

In principle, one can define projectors $(P^{S,A})_{kl}^{ij}$ which act on the tensor product of 2 identical representations $V(\mu)$, and project out the totally symmetric resp. antisymmetric representations. This can be done using the fact that the \hat{R} -matrix discussed in section 1.2.2 has eigenvalues $\pm q^{\frac{1}{2}(c_\lambda - 2c_\mu)}$ with the same sign as classically, and one "just" has to correct the $q^{\frac{1}{2}(c_\lambda - 2c_\mu)}$ -factor. The problem is that this may not give an interesting associative algebra of "totally (anti)symmetric" representations for more than 2 particles, it is probably too restrictive in general. One may hope that something more favorable happens at roots of unity on the space \mathcal{H}_0 .

In this context, we can define an interesting algebra:

$$a_i \overline{a_j} = \pm \overline{a_j} a_i \quad (4.90)$$

with the bar as in (4.70), taking advantage of $\mathcal{U} \ltimes \mathcal{F}$ resp. \mathcal{H} . It is easy to check that this is compatible with the cross-product and our "involution" on \mathcal{H} . Acting on the Fock-vacuum \rangle , this in fact defines totally symmetric resp. antisymmetric 2-particle representations, with a suitable definition of \sqrt{v} . Again, it is not clear if this algebra is interesting for our purpose for more than 2 particles. It would be very desirable to get a better understanding of these issues.

4.4.2 Nonabelian Gauge Fields from Quantum AdS Space

In section 3.2.6, we have found a BRST operator for spin one particles, corresponding to abelian gauge fields. Needless to say that one would also like to consider the nonabelian case. Nonabelian gauge fields are usually described by connections on principal fiber bundles. Therefore one might try to do the same in the q -deformed case, see e.g. [4, 46]. However there is a much simpler, extremely fascinating way to obtain such objects on quantum AdS space (and certain other quantum spaces). It is in fact much simpler than classically. We will only give a rough outline here.

Consider the calculus of differential forms on q -AdS space, which is the same as on quantum Euclidean space, except for the reality structure. As in 2.2.4, the

following observation by Bruno Zumino [65] will be crucial: there exists a "radial" one-form $\omega = \frac{q}{\sqrt{q}-1+\sqrt{q}} \frac{1}{r^2} d(r^2)$ which generates the calculus on quantum Euclidean space, sphere resp. on q-AdS space by

$$[\omega, f]_{\pm} = (\sqrt{q}^{-1} - \sqrt{q})df \equiv \xi df. \quad (4.91)$$

Furthermore on the sphere, it is not possible for $q \neq 1$ to work with "tangential" one-forms only, due to the commutation relations (2.15). Therefore ω must be included in the calculus.

Now consider a matrix of one-forms Ω_j^i (this has nothing to do with the Ω in section 4.4), and write it in the following way:

$$\Omega_j^i = \omega \delta_j^i + \xi B_j^i. \quad (4.92)$$

Physically, we can imagine that this comes from some spontaneous symmetry breaking in the radial fields, which are scalars. One can decompose the one-forms B_j^i into tangential and radial components, e.g. using a Hodge-star operator $*$ which can be defined in a straightforward way. Then

$$\begin{aligned} \Omega &= \omega 1 + \xi(\Phi + A), \quad \text{i.e.} \\ \Omega_j^i &= \omega \delta_j^i + \xi(\phi_j^i \omega + A_{j\mu}^i dy^\mu) \end{aligned} \quad (4.93)$$

with the condition $\omega \wedge *(A_{j,\mu}^i dy^\mu) = 0$, changing notation for the coordinates on AdS space. Furthermore, we can imagine a reality condition like $(\Omega_j^i)^\dagger = -\Omega_i^j$ (without going into details here), so that Ω_j^i corresponds to some Lie algebra, and (4.92) corresponds to $tr\phi = 0$.

Now consider $(\Omega^2)_j^i$, which after a simple calculation using (4.91) becomes

$$\begin{aligned} \Omega^2 &= \xi^2(dB + BB), \quad \text{i.e.} \\ \Omega_k^i \Omega_j^k &= \xi^2(dB_j^i + B_k^i B_j^k). \end{aligned} \quad (4.94)$$

Decomposing it into radial and tangential components, we get

$$\begin{aligned} \Omega^2 &= \xi^2(dA + AA + d\Phi + \Phi A + A\Phi) \\ &= \xi^2(dA + AA + (d\phi + [A, \phi])\omega + o(h)) \end{aligned} \quad (4.95)$$

where $q = e^{2\pi i h}$ as usual. Thus

$$\mathcal{S} = \frac{1}{\xi^2} \int tr(\Omega^2 * \Omega^2) \quad (4.96)$$

gives precisely the Yang–Mills action for a gauge field A coupled to a scalar in the adjoint, like a Higgs field in some GUT models! We do not have to define curvature by hand. This also contains massless BRST ghosts in a nonstandard form, according to section 3.2.6. Moreover, if Ω transforms like

$$\Omega(y) \rightarrow \gamma^{-1}(y)\Omega(y)\gamma(y), \quad (4.97)$$

then the components of Ω transform like

$$\begin{aligned} A &\rightarrow \gamma^{-1}A\gamma + \gamma^{-1}d\gamma + o(h), \quad \text{and} \\ \phi &\rightarrow \gamma^{-1}\phi\gamma + o(h), \end{aligned} \quad (4.98)$$

which are precisely the gauge transformations in a Yang–Mills Theory. Similarly, any trace of polynomials in Ω gives a gauge invariant Lagrangian in the limit $h \rightarrow 0$.

How exactly this fits together with the BRST operator etc. remains to be seen. At the very least, this shows that there is no need to define objects like connections on principal bundles for $q \neq 1$, they arise naturally from the mathematical structure considered. Without elaborating this any further here, we see once again that q – deformation is more than just a ”deformation”, it allows to do things which cannot be done for $q = 1$, and which look very interesting from the point of view of QFT.

Chapter 5

Conclusion

We have shown that many of the essential ingredients of quantum field theories, or more properly quantum theories of elementary particles, have their counterparts in an approach where the classical Poincare group and Minkowski-space are replaced by the quantum Anti-de Sitter group $SO_q(2, 3)$ and a corresponding AdS space, with q a suitable root of unity. First of all, it turned out that there are indeed unitary representations which may describe elementary particles of any spin, with the same low-energy structure as classically. We have defined many-particle representations, which are unitary as well; this is not trivial. In the massless case, it turns out that there is a very natural way to define a Hilbert space using a BRST operator Q , which is an element of the center of $U_q(so(2, 3))$; in the classical case, it has to be defined by hand. Moreover, the same Q works for any spin, and it seems that it can be used to define the many-particle Hilbert space in a similar way. We have also seen how the structure of a nonabelian gauge theory can arise naturally from quantum Anti-de Sitter space; again in the classical case, its description using connections on fiber bundles is quite ad-hoc.

Moreover, everything is manifestly finite in the quantum case. This should not be considered a technicality; it seems that if there is a complete theory of elementary particles, it should be possible to formulate it in a completely well-defined way, without any vague "remedies" under the name of regularization; it should be regular by itself. While it may be too ambitious to attempt finding such a theory, *any* finite version of a 4-dimensional quantum field theory would be very interesting in itself.

Apart from all these mathematical features, one may still wonder if it makes any sense at all to assume that the spacetime we see looking out of the window is noncommutative, i.e. not a classical manifold. I think that this has been answered as

well, by pointing out that if $q = e^{i\pi\hbar}$ is very close to 1 (which it has to be, otherwise there is an obviously clash with observation), the coordinate algebra of quantum Anti-de Sitter space is classical up to corrections involving a length scale $L_0 = \sqrt{\hbar}R$, where R is the curvature radius of AdS space. Thus one should expect it to look like a classical manifold on length scales bigger than L_0 , just like quantum mechanics behaves classically on scales large compared to \hbar . This shows that \hbar must indeed be a *very* small number.

All this can be said in favour of the approach chosen. Nevertheless, we have not yet succeeded in formulating a theory of elementary particles, because we do not know at this time how to define a Hilbert space of identical particles. There may be a satisfactory solution of this problem in the form of a suitable algebra of creation and annihilation operators (in which case we have presented an nice way to define an involution, which is usually a difficult part in the noncommutative case), or perhaps in a completely unexpected way, – or it might be that this is the downfall of the approach. Of course one should hope for the first alternatives, and in view of all the promising features found so far, I cannot believe that this is the end of the story.

Appendix A

Consistency of Involution, Inner Product

Compatibility of (4.70) with cross product. Applying $\overline{(\cdot)}$ to $xa_i = x_1 \triangleright a_i x_2$ in \mathcal{H} , we get

$$\begin{aligned}\Omega a_i \Omega^{-1} \overline{x} &= (\overline{x_2}) \overline{a_i} \pi_i^l(x_1)^* \\ &= \overline{x_2} \Omega a_i \Omega^{-1} \pi_i^i(\overline{x_1}).\end{aligned}\tag{A.1}$$

Multiplying from the left with Ω^{-1} and from the right with Ω , this becomes

$$a_i \Omega^{-1} \overline{x} \Omega \stackrel{!}{=} \Omega^{-1}(\overline{x})_1 \Omega a_i \pi_i^i((\overline{x})_2)\tag{A.2}$$

where $(\overline{x})_{1,2}$ is the Sweedler notation for the coproduct, which becomes using (4.39)

$$\begin{aligned}a_i \tilde{\theta} S^{-1} \overline{x} &\stackrel{!}{=} (\tilde{\theta} S^{-1}(\overline{x})_1)_1 \triangleright a_i (\tilde{\theta} S^{-1}(\overline{x})_1)_2 \pi_i^i((\overline{x})_2) \\ &= (\tilde{\theta} S^{-1}(\overline{x})_{12}) \triangleright a_i \tilde{\theta} S^{-1}(\overline{x})_{11} \pi_i^i((\overline{x})_2) \\ &= a_k \pi_k^l(S^{-1}(\overline{x})_{12}) \pi_i^i((\overline{x})_2) \tilde{\theta} S^{-1}(\overline{x})_{11} \\ &= a_k \pi_k^i((\overline{x})_2 S^{-1}(\overline{x})_{12}) \tilde{\theta} S^{-1}(\overline{x})_{11} \\ &= a_k \pi_k^i(\epsilon(\overline{x}_2)) \tilde{\theta} S^{-1}(\overline{x})_1 \\ &= a_i \tilde{\theta} S^{-1}(\overline{x})\end{aligned}\tag{A.3}$$

as desired, using Lemma 4.3.1 and standard properties of Hopf algebras.

Compatibility of (4.70) with braiding. Applying $\overline{(\cdot)}$ to $a_i b_j = \gamma(\mathcal{R}_2 \triangleright b_j)(\mathcal{R}_1 \triangleright a_i)$, we get

$$\Omega b_j a_i \Omega^{-1} \stackrel{!}{=} \gamma^* \Omega a_k \pi_i^k(\mathcal{R}_1)^* b_l \pi_j^l(\mathcal{R}_2)^* \Omega^{-1},\tag{A.4}$$

or

$$\begin{aligned}
b_j a_i &\stackrel{!}{=} \gamma^* a_k \pi_k^i(\overline{\mathcal{R}_1}) b_l \pi_l^j(\overline{\mathcal{R}_2}) \\
&= \gamma^* a_k \pi_k^i(\mathcal{R}_2^{-1}) b_l \pi_l^j(\mathcal{R}_1^{-1}) \\
&= \gamma^* a_k \pi_i^k(\mathcal{R}_1^{-1}) b_l \pi_j^l(\mathcal{R}_2^{-1}),
\end{aligned} \tag{A.5}$$

using $(\theta \otimes \theta)\mathcal{R} = \mathcal{R}_{21}$ and similarly for $\tilde{\theta}$ in the last line. Multiplying with $\pi_s^j(\mathcal{R}_b) \pi_t^i(\mathcal{R}_a)$, we get

$$b_j \pi_s^j(\mathcal{R}_b) a_i \pi_t^i(\mathcal{R}_a) \stackrel{!}{=} \gamma^* a_t b_s, \tag{A.6}$$

which is just the braiding algebra provided $\gamma^* = \gamma^{-1}$.

Equality of the inner products (4.78) Using $\Delta(\Omega) = \Delta(\sqrt{v^{-1}})\mathcal{R}^{-1}(\tilde{w} \otimes \tilde{w})$ and $\langle f(a)x = \langle (S^{-1}x) \triangleright f(a) \rangle$ we can write the lhs of (4.78) explicitly as

$$\begin{aligned}
(a_i b_j \dots, a_k b_l \dots) &= \langle ((\sqrt{v^{-1}})_1 \mathcal{R}_1^{-1} \tilde{w}) \triangleright (\dots b_j a_i) (\sqrt{v^{-1}})_2 \mathcal{R}_2^{-1} \tilde{w} \tilde{w}^{-1} \sqrt{v} a_k b_l \dots \rangle \\
&= \langle ((\sqrt{v^{-1}})_1 \mathcal{R}_1^{-1} \tilde{w}) \triangleright (\dots b_j a_i) (\sqrt{v^{-1}})_2 \mathcal{R}_2^{-1} \sqrt{v} a_k b_l \dots \rangle \\
&= \langle (S^{-1}((\sqrt{v^{-1}})_2 \mathcal{R}_2^{-1} \sqrt{v}) (\sqrt{v^{-1}})_1 \mathcal{R}_1^{-1} \tilde{w}) \triangleright (\dots b_j a_i) a_k b_l \dots \rangle \\
&= \langle (S^{-1}(\mathcal{R}_2^{-1} \sqrt{v}) \epsilon(\sqrt{v^{-1}}) \mathcal{R}_1^{-1} \tilde{w}) \triangleright (\dots b_j a_i) a_k b_l \dots \rangle \\
&= \langle (\sqrt{v} \mathcal{R}_2 S^2 \mathcal{R}_1 \tilde{w}) \triangleright (\dots b_j a_i) a_k b_l \dots \rangle \\
&= \langle (\sqrt{v} q^{-2\bar{\rho}} v^{-1} \tilde{w}) \triangleright (\dots b_j a_i) a_k b_l \dots \rangle \\
&= \langle (\sqrt{v} (-1)^E v^{-1} w) \triangleright (\dots b_j a_i) a_k b_l \dots \rangle \\
&= \langle (v^{-1} w) \triangleright (\dots b_j a_i) \sqrt{v} (-1)^E a_k b_l \dots \rangle
\end{aligned} \tag{A.7}$$

using (1.29) ff., (1.54), (4.38) and $\langle (x \triangleright a) b \rangle = \langle a(Sx) b \rangle$, $\epsilon(\sqrt{v^{-1}}) = 1$, which is easy to see. In particular, with (4.52) we can verify that

$$\begin{aligned}
(a_i, a_j) \equiv \langle \overline{a_i} a_j \rangle &= \langle (v^{-1} w \triangleright a_i) \sqrt{v} (-1)^E a_j \rangle \\
&= g_a q^{\frac{1}{2}c_a} (-1)^{E_j} (-1)^{f_a} g_{kj} \pi_i^k(w) \\
&= g_a q^{c_a} (-1)^{E_j + f_a} \pi_k^j((-1)^E w) \pi_i^k(w) \\
&= g_a \delta_j^i
\end{aligned} \tag{A.8}$$

is a (positive definite) inner product on the unitary representations a_i ; it had to come out this way, since we always use the orthogonal basis of Lemma 4.3.1.

Using the first line of (A.8) and $\Delta_{(n)}(v^{-1} w) = (v^{-1} w \otimes \dots \otimes v^{-1} w) \mathcal{R}_{(n)}$, we can continue (A.7) as

$$\begin{aligned}
rhs &= \langle \dots (v^{-1} w) \triangleright b_s \pi_j^s(\mathcal{R}_{n-1}) (v^{-1} w) \triangleright a_t \pi_i^t(\mathcal{R}_n) \sqrt{v} (-1)^E a_k b_l \dots \rangle \\
&= q^{\frac{1}{2}(c_a + c_b + \dots)} \pi_i^t(\mathcal{R}_n) \pi_j^s(\mathcal{R}_{n-1}) \dots (a_t \otimes b_s \otimes \dots, \sqrt{v} \triangleright (a_k \otimes b_l \otimes \dots))_{\otimes}
\end{aligned} \tag{A.9}$$

where $\mathcal{R}_{12\dots}$ are the components of $\mathcal{R}_{(n)}$ assuming there are n factors, and using $\Delta(-1)^E = (-1)^E \otimes \dots \otimes (-1)^E$. Now notice that (A.8) and (1.63) imply

$$\begin{aligned}
\pi_i^t(\mathcal{R}_2)\pi_j^s(\mathcal{R}_1)(a_t \otimes b_s, a_k \otimes b_l)_\otimes &= g_a g_b \pi_i^k(\mathcal{R}_2)\pi_j^l(\mathcal{R}_1) \\
&= (a_i \otimes b_j, a_t \otimes b_s)_\otimes \pi_k^t(\mathcal{R}_1)\pi_l^s(\mathcal{R}_2) \\
&= (a_i \otimes b_j, a_k \otimes b_l)_\mathcal{R}
\end{aligned} \tag{A.10}$$

and similarly for several factors, so (A.9) is nothing but

$$\begin{aligned}
rhs &= c_V(a_i \otimes b_j \otimes \dots, \sqrt{v} \triangleright (a_k \otimes b_l \otimes \dots))_\mathcal{R} \\
&= (a_i \otimes b_j \otimes \dots, a_k \otimes b_l \otimes \dots)_H
\end{aligned} \tag{A.11}$$

as in (4.29).

Appendix B

On the Weyl Element w

In this appendix, we want to give some explanations to the remarkable formulas (4.34), (4.35) and (4.37).

As mentioned before, the braid group action (1.46) on \mathcal{U} can in fact be extended to a braid group action T_i on any representation, with explicit formulas in terms of (infinite) sums of generators of \mathcal{U} [39], see also [25]. Then one can consider the element of the braid group corresponding to the longest element of the Weyl group, call it T . It was shown in [33, 34] that these T_i and therefore T can be implemented by a conjugation with w_i resp. w , which are elements of an extension of \mathcal{U} , and that w satisfies (4.34) with suitable definitions. Unfortunately, this requires complicated calculations.

To get some faith in these formulas, we want to point out first that if (4.34)

$$\Delta(w) = \mathcal{R}^{-1}w \otimes w \tag{B.1}$$

holds on the tensor product of 2 fundamental representations, it holds for any representations. This can be seen inductively: if (B.1) holds on $V_i \otimes V_j$, then it also holds on the representations in $V_1 \otimes V_2 \otimes V_3$, since the rhs of

$$(\text{id} \otimes \Delta)\Delta(w) \stackrel{!}{=} \mathcal{R}_{1,(23)}^{-1}(w \otimes w) \tag{B.2}$$

acting on $V_1 \otimes (V_2 \otimes V_3)$ agrees with the rhs of

$$(\Delta \otimes \text{id})\Delta(w) \stackrel{!}{=} \mathcal{R}_{(12),3}^{-1}(w \otimes w) \tag{B.3}$$

acting on $(V_1 \otimes V_2) \otimes V_3$, by the first statement of Lemma 1.1.1. Thus the lhs agree as they should.

Furthermore, since the action of w on any given (finite-dimensional) representation can be expressed in terms of generators of \mathcal{U} , it satisfies $\Delta'(w) = \mathcal{R}\Delta(w)\mathcal{R}^{-1}$, and together with (B.1) this implies $\Delta(w) = w \otimes w\mathcal{R}_{21}$.

The action of w_i resp. w on the fundamental representations can be found explicitly, see [49]. For real groups, it is essentially the invariant metric g_{ij} .

(4.35) together with the statement that $\varepsilon = \pm 1$ on any irrep now follows easily, since $\varepsilon \equiv v^{-1}w^2$ is grouplike as already pointed out, i.e. $\Delta(\varepsilon) = \varepsilon \otimes \varepsilon$, and it only remains to check that ε is ± 1 on the fundamental representations, where it agrees with the classical limit. This also follows from consistency with the tensor product.

A proof of $wxw^{-1} = \theta Sx$ for real groups. Here we only consider the case of "real" groups, which are all except A_n , D_n and E_6 , i.e. those for which the Dynkin diagram does not have an automorphism.

From (4.34), (1.49) and (1.29), we know that

$$(w \otimes w)\mathcal{R}(w^{-1} \otimes w^{-1}) = \mathcal{R}_{21} = (\theta S \otimes \theta S)\mathcal{R} \quad (\text{B.4})$$

Furthermore, it is known that the first terms of the expansion of \mathcal{R} in powers of X_i^\pm are

$$\mathcal{R} = q^{\sum (\alpha^{-1})_{ij} h_i \otimes h_j} \left(1 + \sum_i c_i(q) q^{\frac{1}{2}h_i} X_i^+ \otimes q^{-\frac{1}{2}h_i} X_i^- + \dots \right) \quad (\text{B.5})$$

Now $wX_i^\pm w^{-1}$ certainly has the form $\mathcal{U}^0 X_i^\mp$, as can be seen either from the explicit formulas in [33, 34] or from the cross product algebra (4.62). On the Cartan subalgebra, w acts as classically. Thus by the uniqueness of \mathcal{R} [32, 16], (B.4) implies that

$$wX_i^\pm w^{-1} = a_i \theta S(X_i^\mp) \quad (\text{B.6})$$

and

$$wX_i^\pm w^{-1} = a_i^{-1} \theta S(X_i^\mp) \quad (\text{B.7})$$

with a constant a_i . Notice that if there exists an automorphism of the Dynkin diagram, then this would only follow up to a corresponding permutation of the simple root vectors. The constants a_i can be eliminated by a redefinition $w \rightarrow wq^{b_i H_i}$ with suitable constants b_i , using the fact that the Cartan matrix is non-singular. This new w now satisfies all the equations discussed.

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